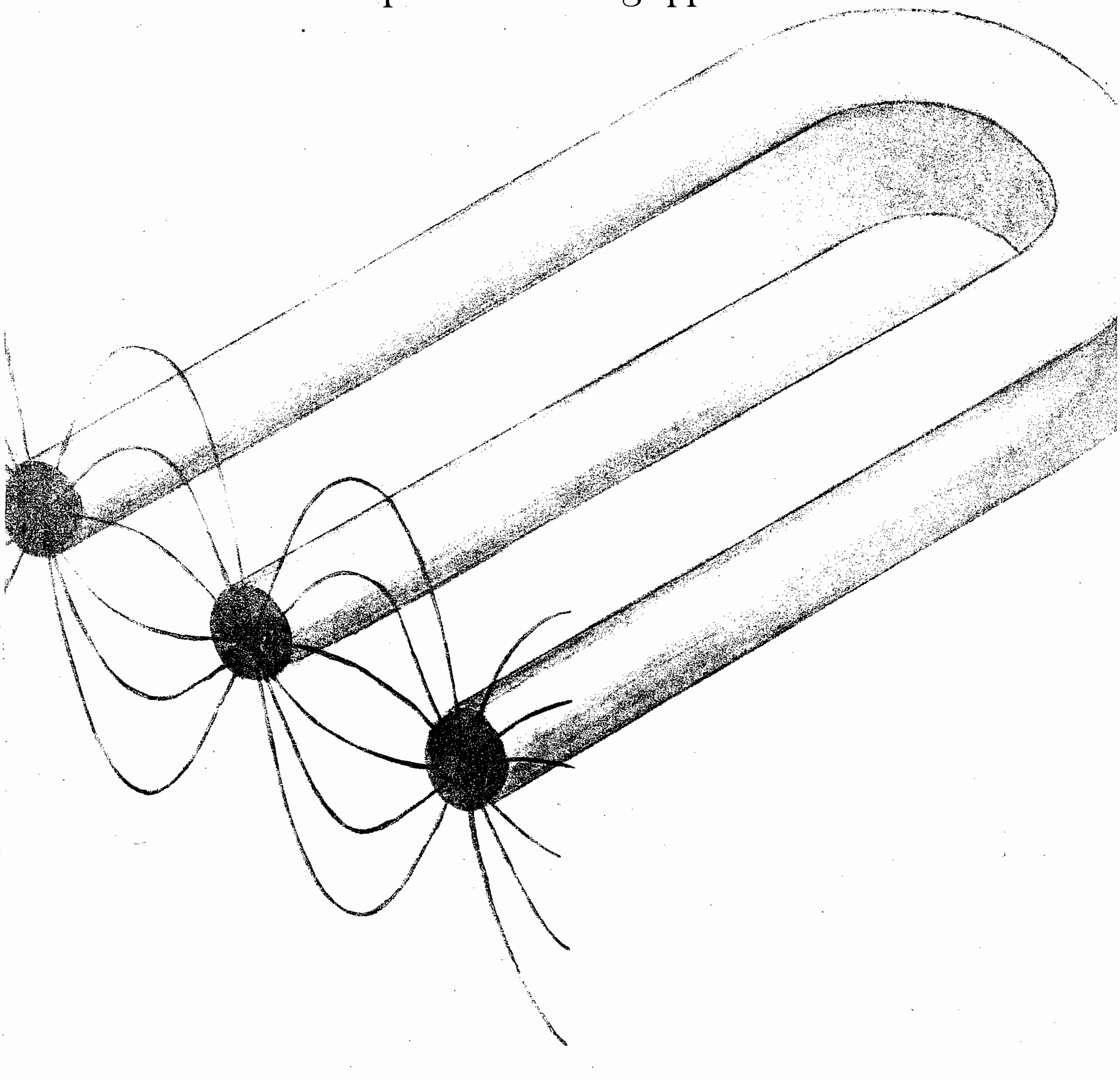
Markus Zahn

instructor's manual

ELECTROMAGNETIC FIELD THEORY

a problem solving approach



Instructor's Manual

to accompany

ELECTROMAGNETIC FIELD THEORY:
A PROBLEM SOLVING APPROACH

by

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PREFACE

To follow the problem solving approach of the text, there are a large number of exercise problems at the end of each chapter grouped together by topic and thus text section number. Using this manual as an aid together with the problem statements in the text, an instructor can easily choose those problems which reinforce the classroom materials.

The author would be appreciative of any corrections or other comments to this manual or to the text.

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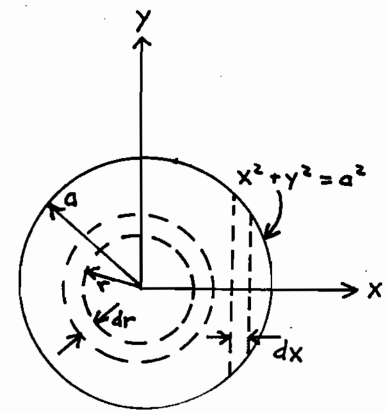
CHAPTER 1
REVIEW OF VECTOR ANALYSIS

Section 1.1



1. a) Area $= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} dxdy$

$$= 2 \int_{x=-a}^{a} \sqrt{a^2 - x^2}\, dx$$

$$= [x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a}]\Big|_{x=-a}^{a} = \pi a^2$$

b) Area $= \int_{r=0}^{a} \int_{\phi=0}^{2\pi} rdrd\phi$

$$= 2\pi \frac{r^2}{2}\Big|_{r=0}^{a} = \pi a^2 \qquad \text{[Easier than (a)]}$$

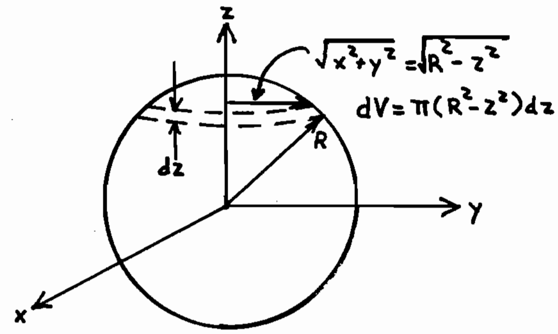
2. a) Volume $= \int_{z=-R}^{R} \int_{y=-\sqrt{R^2-z^2}}^{+\sqrt{R^2-z^2}} \int_{x=-\sqrt{R^2-y^2-z^2}}^{+\sqrt{R^2-y^2-z^2}} dxdydz$

$$= 2 \int_{z=-R}^{R} \int_{y=-\sqrt{R^2-z^2}}^{+\sqrt{R^2-z^2}} \sqrt{R^2 - y^2 - z^2}\; dydz$$

$$= \int_{z=-R}^{R} \left[y\sqrt{R^2 - y^2 - z^2} + (R^2 - z^2)\sin^{-1} \frac{y}{\sqrt{R^2 - z^2}} \right]\Bigg|_{y=-\sqrt{R^2-z^2}}^{+\sqrt{R^2-z^2}} dz$$

$$= \pi \int_{z=-R}^{R} (R^2 - z^2)\, dz$$

$$= \pi(R^2 z - \frac{z^3}{3})\Big|_{z=-R}^{R} = \frac{4}{3}\pi R^3$$



b) $$\text{Volume} = \int_{\phi=0}^{2\pi}\int_{z=-R}^{R}\int_{r=0}^{\sqrt{R^2-z^2}} r\,dr\,d\phi\,dz$$

$$= \int_{z=-R}^{R} \pi(R^2 - z^2)\,dz$$

$$= \frac{4}{3}\pi R^3$$

c) $$\text{Volume} = \int_{r=0}^{R}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^2 \sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_{r=0}^{R} -2\pi \cos\theta\Big|_{\theta=0}^{\pi} r^2 dr$$

$$= \int_{r=0}^{R} 4\pi r^2 dr$$

$$= \frac{4}{3}\pi r^3\Big|_{r=0}^{R} = \frac{4}{3}\pi R^3$$ [Easier than (a) or (b)]

Section 1.2

3. a) $\bar{A} + \bar{B} = 6\bar{i}_x - 2\bar{i}_y - 6\bar{i}_z$

$\bar{A} - \bar{B} = 6\bar{i}_y + 4\bar{i}_z$

$\bar{B} + \bar{C} = 4\bar{i}_x - 5\bar{i}_y - 4\bar{i}_z$

$\bar{B} - \bar{C} = 2\bar{i}_x - 3\bar{i}_y - 6\bar{i}_z$

$\bar{A} + \bar{C} = 4\bar{i}_x + \bar{i}_y$

$\bar{A} - \bar{C} = 2\bar{i}_x + 3\bar{i}_y - 2\bar{i}_z$

b) $\bar{A} \cdot \bar{B} = 9 - 8 + 5 = 6$

$\bar{B} \cdot \bar{C} = 3 + 4 - 5 = 2$

$\bar{A} \cdot \bar{C} = 3 - 2 - 1 = 0$

c) $$\overline{A}\times\overline{B} = \det\begin{vmatrix} \overline{i}_x & \overline{i}_y & \overline{i}_z \\ 3 & 2 & -1 \\ 3 & -4 & -5 \end{vmatrix} = -14\overline{i}_x + 12\overline{i}_y - 18\overline{i}_z$$

$$\overline{B}\times\overline{C} = \det\begin{vmatrix} \overline{i}_x & \overline{i}_y & \overline{i}_z \\ 3 & -4 & -5 \\ 1 & -1 & +1 \end{vmatrix} = -9\overline{i}_x - 8\overline{i}_y + \overline{i}_z$$

$$\overline{A}\times\overline{C} = \det\begin{vmatrix} \overline{i}_x & \overline{i}_y & \overline{i}_z \\ 3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \overline{i}_x - 4\overline{i}_y - 5\overline{i}_z$$

d) $$\left.\begin{aligned} (\overline{A}\times\overline{B})\cdot\overline{C} &= (-14\overline{i}_x + 12\overline{i}_y - 18\overline{i}_z)\cdot(\overline{i}_x - \overline{i}_y + \overline{i}_z) = -44 \\ \overline{A}\cdot(\overline{B}\times\overline{C}) &= (3\overline{i}_x + 2\overline{i}_y - \overline{i}_z)\cdot(-9\overline{i}_x - 8\overline{i}_y + \overline{i}_z) = -44 \end{aligned}\right\}\ \text{equal}$$

e) $$\overline{A}\times(\overline{B}\times\overline{C}) = \det\begin{vmatrix} \overline{i}_x & \overline{i}_y & \overline{i}_z \\ 3 & 2 & -1 \\ -9 & -8 & 1 \end{vmatrix} = -6\overline{i}_x + 6\overline{i}_y - 6\overline{i}_z$$

$$\overline{B}(\overline{A}\cdot\overline{C}) - \overline{C}(\overline{A}\cdot\overline{B}) = -6\overline{C} = -6(\overline{i}_x - \overline{i}_y + \overline{i}_z) \quad \text{equal}$$

f) $$\cos\theta_{AC} = \frac{\overline{A}\cdot\overline{C}}{AC} = 0 \rightarrow \theta_{AC} = 90^\circ$$

$$\cos\theta_{B,\overline{A}\times\overline{C}} = \frac{\overline{B}\cdot(\overline{A}\times\overline{C})}{B\,|\overline{A}\times\overline{C}|} = \frac{44}{\sqrt{50(42)}} \approx .96 \rightarrow \theta_{B,\overline{A}\times\overline{C}} = 16.2^\circ$$

4. $\overline{A} + \overline{B} = -\overline{i}_x + 5\overline{i}_y - 4\overline{i}_z$

$\overline{A} - \overline{B} = 3\overline{i}_x - \overline{i}_y - 2\overline{i}_z$

Add: $2\overline{A} = 2\overline{i}_x + 4\overline{i}_y - 6\overline{i}_z \rightarrow \overline{A} = \overline{i}_x + 2\overline{i}_y - 3\overline{i}_z$

Subtract: $2\overline{B} = -4\overline{i}_x + 6\overline{i}_y - 2\overline{i}_z \rightarrow \overline{B} = -2\overline{i}_x + 3\overline{i}_y - \overline{i}_z$

5. a) $\overline{B}_{||} = \alpha\overline{A}$, $\overline{A}\cdot\overline{B} = AB_{||} = \alpha\overline{A}\cdot\overline{A} \rightarrow \alpha = \dfrac{\overline{A}\cdot\overline{B}}{\overline{A}\cdot\overline{A}}$

b) $\overline{B}_{||} = \dfrac{\overline{A}\cdot\overline{B}}{\overline{A}\cdot\overline{A}}\overline{A} = \dfrac{(-12)}{6}(\overline{i}_x - 2\overline{i}_y + \overline{i}_z) = 2(-\overline{i}_x + 2\overline{i}_y - \overline{i}_z)$

$\overline{B}_{\perp} = \overline{B} - \overline{B}_{||} = 5\overline{i}_x + \overline{i}_y - 3\overline{i}_z$

6. $\overline{A}\cdot\overline{B} = -36$, $\overline{A}\cdot\overline{C} = 0$ ($\overline{A}$ and $\overline{C}$ perpendicular), $\overline{B}\cdot\overline{C} = 0$ ($\overline{B}$ and $\overline{C}$ perpendicular)

$$\cos\theta_{AB} = \frac{\overline{A}\cdot\overline{B}}{AB} = \frac{-36}{\sqrt{24}\,\sqrt{54}} = -1 \rightarrow \theta_{A,B} = 180°$$

7. a) $\overline{A}\cdot\overline{B} = -75$

b)
$$\overline{A}\times\overline{B} = \det\begin{vmatrix} \overline{i}_x & \overline{i}_y & \overline{i}_z \\ 3 & 4 & 0 \\ 7 & -24 & 0 \end{vmatrix} = -100\overline{i}_z$$

c) $\cos\theta_{AB} = \dfrac{\overline{A}\cdot\overline{B}}{AB} = \dfrac{-75}{5(25)} = -.6 \rightarrow \theta_{AB} = 126.87°$

8. $$\cos\alpha = \frac{\overline{A}\cdot\overline{i}_x}{A} = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$$\cos\beta = \frac{\overline{A}\cdot\overline{i}_y}{A} = \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$$\cos\gamma = \frac{\overline{A}\cdot\overline{i}_z}{A} = \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{A_x^2 + A_y^2 + A_z^2}{A_x^2 + A_y^2 + A_z^2} = 1$$

9. a) $\overline{C}\cdot\overline{C} = (\overline{B}-\overline{A})\cdot(\overline{B}-\overline{A}) = \overline{B}\cdot\overline{B} - 2\overline{A}\cdot\overline{B} + \overline{A}\cdot\overline{A} \rightarrow C^2 = A^2 + B^2 - 2AB\cos\theta_c$

b) $\overline{B}\times\overline{A} = (\overline{C}+\overline{A})\times\overline{A} = \overline{C}\times\overline{A}$

$$|\overline{B}\times\overline{A}| = AB\sin\theta_c = AC\sin(\pi - \theta_b) = AC\sin\theta_b$$

$$\frac{\sin\theta_b}{B} = \frac{\sin\theta_c}{C}$$

$$\overline{B}\times\overline{C} = \overline{B}\times(\overline{B}-\overline{A}) = \overline{A}\times\overline{B}$$

$$|\overline{B}\times\overline{C}| = BC\sin\theta_a = |\overline{A}\times\overline{B}| = AB\sin\theta_c$$

$$\frac{\sin\theta_c}{C} = \frac{\sin\theta_b}{B} = \frac{\sin\theta_a}{A}$$

10. a)

$$(\overline{A}\times\overline{B})\cdot\overline{C} = \det\begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$(\overline{B}\times\overline{C})\cdot\overline{A} = \det\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (\overline{A}\times\overline{B})\cdot\overline{C}$$

$$(\overline{C}\times\overline{A})\cdot\overline{B} = \det\begin{vmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \\ A_x & A_y & A_z \end{vmatrix} = (\overline{A}\times\overline{B})\cdot\overline{C}$$

b) The vector $\overline{A} \times \overline{B}$ is normal to the plane formed by the vectors $\overline{A}$ and $\overline{B}$ while its magnitude $AB\sin\theta_{ab}$ gives the area of the parallelogram formed by $\overline{A}$ and $\overline{B}$. The component of $\overline{C}$ in the direction of $\overline{A} \times \overline{B}$ defines the altitude of the parallelepiped formed by $\overline{A}$, $\overline{B}$, and $\overline{C}$, so that the enclosed volume is

$$V = |(\overline{A} \times \overline{B})\cdot\overline{C}|$$

c)
$$V = \overline{A} \times \overline{B} \cdot \overline{C} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & 0 \end{vmatrix} = 4$$

d)
$$\begin{aligned}
\overline{A} \times (\overline{B} \times \overline{C}) &= (A_x\overline{i}_x + A_y\overline{i}_y + A_z\overline{i}_z) \\
&\quad \times [(B_yC_z - B_zC_y)\overline{i}_x + (B_zC_x - B_xC_z)\overline{i}_y + (B_xC_y - B_yC_x)\overline{i}_z] \\
&= \overline{i}_x[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)] \\
&\quad + \overline{i}_y[-A_x(B_xC_y - B_yC_x) + A_z(B_yC_z - B_zC_y)] \\
&\quad + \overline{i}_z[A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)] \\
&= [B_x\overline{i}_x + B_y\overline{i}_y + B_z\overline{i}_z][A_xC_x + A_yC_y + A_zC_z] \\
&\quad - [C_x\overline{i}_x + C_y\overline{i}_y + C_z\overline{i}_z][A_xB_x + A_yB_y + A_zC_z] \\
&= \overline{B}(\overline{A}\cdot\overline{C}) - \overline{C}(\overline{A}\cdot\overline{B})
\end{aligned}$$

11. a) $\overline{A} = A\cos\theta\overline{i}_x + A\sin\theta\overline{i}_y$

$\overline{B} = B\cos\phi\overline{i}_x - B\sin\phi\overline{i}_y$

b) $\overline{A}\cdot\overline{B} = AB\cos(\theta + \phi) = AB\cos\theta \cos\phi - AB\sin\theta \sin\phi$

$$\rightarrow \cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$$

$$|\overline{A} \times \overline{B}| = AB\sin(\theta + \phi) = |-AB(\cos\theta \sin\phi + \sin\theta \cos\phi)\overline{i}_z|$$

$$\rightarrow \sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

Section 1.3

12. a) $\nabla f = \dfrac{\partial f}{\partial x}\overline{i}_x + \dfrac{\partial f}{\partial y}\overline{i}_y + \dfrac{\partial f}{\partial z}\overline{i}_z$

$= (az + 3bx^2y)\overline{i}_x + bx^3\overline{i}_y + ax\overline{i}_z$

b) $$\nabla f = \frac{\partial f}{\partial r}\bar{i}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\bar{i}_\phi + \frac{\partial f}{\partial z}\bar{i}_z$$

$$= \left(-\frac{a}{r^2}\sin\phi + bz^2\cos 3\phi\right)\bar{i}_r + \left(\frac{a}{r^2}\cos\phi - 3bz^2\sin 3\phi\right)\bar{i}_\phi + 2brz\cos 3\phi\,\bar{i}_z$$

c) $$\nabla f = \frac{\partial f}{\partial r}\bar{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\bar{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\bar{i}_\phi$$

$$= \left(a\cos\theta - \frac{2b}{r^3}\sin\phi\right)\bar{i}_r - a\sin\theta\,\bar{i}_\theta + \frac{b}{r^3}\frac{\cos\phi}{\sin\theta}\bar{i}_\phi$$

13. $f = r\sin\phi \rightarrow \nabla f = \sin\phi\,\bar{i}_r + \cos\phi\,\bar{i}_\phi = \bar{i}_y$

$$\int_1 \nabla f\cdot\overline{d\ell} = \int_{-a}^{+a} dy = 2a$$

$$\int_2 \nabla f\cdot\overline{d\ell} = \int_{-\pi/2}^{+\pi/2} a\cos\phi\, d\phi = a\sin\phi\Big|_{-\pi/2}^{+\pi/2} = 2a$$

$$\int_3 \nabla f\cdot\overline{d\ell} = 2a$$

Section 1.4

14. a) $$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3$$

$\left(\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial (r^3)}{\partial r} = 3 \quad \text{spherical coordinates}\right)$

b) $$\nabla\cdot\bar{A} = y^2z^3 + 2xyz^3 + 3xy^2z^2$$

c) $$\nabla\cdot\bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= 2\cos\phi + \frac{\sin\phi}{r}$$

d) $$\nabla\cdot\overline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$= 4r\sin\theta\cos\phi + 2r\cos\theta\cos\phi - r\sin\phi$$

15. a) $$\int_V \nabla\cdot\overline{A}\, dV = \oint_S \overline{A}\cdot\overline{dS} \qquad \text{(Divergence Theorem)}$$

Let $\overline{A} = \overline{i}\, f \rightarrow \nabla\cdot\overline{A} = (\overline{i}\cdot\nabla)f + f\, \cancelto{0}{\nabla\cdot\overline{i}}$

$$\overline{i}\cdot\int \nabla f dV = \overline{i}\cdot\oint_S f\,\overline{dS} \rightarrow \int_V \nabla f dV = \oint_S f\,\overline{dS}$$

b) Let $\overline{A} = \overline{i}\times\overline{F} \rightarrow \nabla\cdot\overline{A} = \nabla\cdot(\overline{i}\times\overline{F}) = \overline{F}\cdot(\cancelto{0}{\nabla\times\overline{i}}) - \overline{i}\cdot\nabla\times\overline{F}$ (See Prob. 24c)

$$\int_V \nabla\cdot\overline{A}dV = -\overline{i}\cdot\int_V \nabla\times\overline{F}\, dV$$

$$= \oint_S \overline{A}\cdot\overline{dS}$$

$$\left.\begin{aligned} &= \oint_S (\overline{i}\times\overline{F})\cdot\overline{dS} \\ &= \overline{i}\cdot\oint_S \overline{F}\times\overline{dS}\end{aligned}\right\} \text{(See Prob. 10a)}$$

$$\int_V \nabla\times\overline{F}\, dV = -\oint_S \overline{F}\times\overline{dS}$$

c) Let $f = 1$ in (a) $\rightarrow \nabla f = 0$

$$\oint \overline{dS} = 0$$

d) $\overline{n} = \overline{i}_r = \sin\theta\cos\phi\,\overline{i}_x + \sin\theta\sin\phi\,\overline{i}_y + \cos\theta\,\overline{i}_z$

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \overline{i}_r\, R^2\sin\theta d\theta d\phi = R^2\left[\overline{i}_x(0) + \overline{i}_y(0) - \frac{2\pi R^2}{4}\cos 2\theta\Big|_{\theta=0}^{\pi}\overline{i}_z\right] = 0$$

16. $\nabla\cdot(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$
$\nabla\cdot(g\nabla f) = g\nabla^2 f + \nabla f\cdot\nabla g$ } (See Prob. 24c)

Subtract

$$\nabla\cdot(f\nabla g)-\nabla\cdot(g\nabla f) = f\nabla^2 g - g\nabla^2 f$$

$$\int_V \nabla\cdot[f\nabla g - g\nabla f]dV = \oint_S [f\nabla g - g\nabla f]\cdot\overline{dS} = \int_V [f\nabla^2 g - g\nabla^2 f]dV$$

17. a) $\overline{dS}_1 = -dxdy\overline{i}_z$

$\overline{dS}_2 = -dxdz\overline{i}_y$

$\overline{dS}_3 = -dydz\overline{i}_x$

The equation of the plane surface 4 is of the form

$$x + Ay + Bz = C$$

where the constants A, B, and C are found by specifying three points in the plane.

$(a,0,0)$

$(0,b,0) \rightarrow f = x + \frac{a}{b}y + \frac{a}{c}z - a = 0$

$(0,0,c)$

The unit normal to the surface is found from ∇f as

$$\overline{n} = \frac{\nabla f}{|\nabla f|} = \frac{\overline{i}_x + \frac{a}{b}\overline{i}_y + \frac{a}{c}\overline{i}_z}{\sqrt{1 + (\frac{a}{b})^2 + (\frac{a}{c})^2}}$$

The differential surface area is then

$$dS_4 = \frac{dxdy\ \overline{n}}{\cos\gamma}$$

where γ is the angle between surface 4 and the xy plane

$$\cos\gamma = \overline{n}\cdot\overline{i}_z = \frac{\frac{a}{c}}{\sqrt{1 + (\frac{a}{b})^2 + (\frac{a}{c})^2}} \rightarrow \overline{dS}_4 = \frac{c}{a}dxdy[\overline{i}_x + \frac{a}{b}\overline{i}_y + \frac{a}{c}\overline{i}_z]$$

b) $\overline{A} = r\overline{i}_r = x\overline{i}_x + y\overline{i}_y + z\overline{i}_z$

$$\Phi = \oint_S \overline{A}\cdot\overline{dS}$$

$$= -\int_1 A_z(z=0)\,dxdy - \int_2 A_y(y=0)\,dxdz - \int_3 A_x(x=0)\,dydz$$

$$+ \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} \overline{A}\cdot\overline{dS}_4$$

$$z = -\frac{c}{a}\left(x + \frac{a}{b}y - a\right)$$

$$= \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} [\underbrace{x + \frac{a}{b}y + \frac{c}{a}z}_{a}]\,\frac{c}{a}\,dxdy$$

$$= \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} c\,dxdy$$

$$= \int_{x=0}^{a} cb\left[1 - \frac{x}{a}\right]dx$$

$$= \frac{abc}{2}$$

c) $\nabla\cdot\overline{A} = 3$

$$\Phi = \int_V \nabla\cdot\overline{A}\,dV = 3\,\underbrace{\frac{1}{3}\,\frac{abc}{2}}_{\substack{\text{volume}\\ \text{of}\\ \text{pyramid}}} = \frac{abc}{2}$$

Section 1.5

18. a) $\nabla\times\overline{A} = \overline{i}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \overline{i}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \overline{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$

$$= (x - y^2)\overline{i}_x - y\overline{i}_y - x^2\overline{i}_z$$

b) $\nabla \times \overline{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\overline{i}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\overline{i}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\overline{i}_z$

$$= -\sin\phi\,\overline{i}_r + \left(\frac{\sin\phi}{r} - \cos\phi\right)\overline{i}_\phi - \frac{z}{r^2}\cos\phi\,\overline{i}_z$$

c) $\nabla \times \overline{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\overline{i}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\overline{i}_\theta$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\overline{i}_\phi$$

$$= -\frac{\cos\phi\cot\theta}{r^3}\overline{i}_r - r\sin\phi\,\overline{i}_\theta - \left[\frac{\cos\theta\sin\phi}{r^3} + r\cos\theta\cos\phi\right]\overline{i}_\phi$$

19. Let $\overline{A} = \overline{i}f$, $\nabla \times \overline{A} = \nabla f \times \overline{i} + f\,\cancelto{0}{\nabla \times \overline{i}}$ (See Prob. 24f)

$$\oint_L \overline{A}\cdot\overline{d\ell} = \overline{i}\cdot\oint_L f\overline{d\ell} = \int_S (\nabla \times \overline{A})\cdot\overline{dS} = -\int_S \underbrace{(\overline{i} \times \nabla f)\cdot\overline{dS} = -\overline{i}\cdot\int_S \nabla f \times \overline{dS}}_{\text{(See Prob. 10a)}}$$

$$\oint_L f\overline{d\ell} = -\int_S \nabla f \times \overline{dS}$$

20. $\overline{A} = (x + a)(y + b)(z + c)\overline{i}_x$

$$\nabla \times \overline{A} = -\frac{\partial A_x}{\partial y}\overline{i}_z + \frac{\partial A_x}{\partial z}\overline{i}_y$$

$$= (x + a)[-(z + c)\overline{i}_z + (y + b)\overline{i}_y]$$

$$\oint_L \overline{A}\cdot\overline{d\ell} = \int_{\substack{x=0\\y=y_1\\z=0}}^{x_1} A_x dx + \int_{\substack{x=x_1\\y=y_2\\z=0}}^{0} A_x dx$$

$$= \frac{(x + a)^2}{2}(y_1 + b)c\Big|_{x=0}^{x_1} + \frac{(x + a)^2}{2}(y_2 + b)c\Big|_{x=x_1}^{0}$$

$$= \frac{c}{2}\{(y_1 + b)[(x_1 + a)^2 - a^2] - (y_2 + b)[(x_1 + a)^2 - a^2]\}$$

$$= \frac{c}{2}[(x_1 + a)^2 - a^2][y_1 - y_2]$$

$$\int \nabla \times \bar{A}\cdot\overline{dS} = \iint (\nabla \times \bar{A})_z\,dxdy$$

flat surface
at z=0

$$= \int_{x=0}^{x_1}\int_{y=y_1}^{y_2} -c(x+a)dxdy = -\frac{c}{2}(y_2 - y_1)[(x_1+a)^2 - a^2]$$

$$\underset{\substack{\text{rectangular}\\ \text{cylinder}}}{\int} \nabla \times \bar{A}\cdot\overline{dS} = \int_{\substack{y=y_2\\ \text{surface}}} (\nabla \times \bar{A})_y dxdz - \int_{\substack{y=y_1\\ \text{surface}}} (\nabla \times \bar{A})_y dxdy + \int_{\substack{x=x_1\\ \text{surface}}} \cancelto{0}{(\nabla \times \bar{A})_x} dydz$$

$$- \int_{\substack{x=0\\ \text{surface}}} \cancelto{0}{(\nabla \times \bar{A})_x} dydz + \int_{\substack{z=z_1\\ \text{surface}}} (\nabla \times \bar{A})_z dxdy$$

$$= \int_0^{z_1}\int_{x=0}^{x_1} (x+a)(y_2+b)dxdz - \int_0^{z_1}\int_{x=0}^{x_1} (x+a)(y_1+b)dxdz$$

$$- \int_0^{x_1}\int_{y=y_1}^{y_2} (z_1 + c)(x+a)dxdy$$

$$= (y_2+b)\frac{z_1}{2}[(x_1+a)^2 - a^2] - (y_1+b)\frac{z_1}{2}[(x_1+a)^2 - a^2]$$

$$- \frac{(z_1+c)(y_2-y_1)}{2}[(x_1+a)^2 - a^2]$$

$$= -\frac{c}{2}[(x_1+a)^2 - a^2][y_2 - y_1]$$

21. $f = \dfrac{x^2 \ln y}{y}$

$$\frac{\partial f}{\partial x} = \frac{2x\ln y}{y}\ ; \quad \frac{\partial f}{\partial y} = \frac{x^2}{y^2}[1 - \ln y]$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{2x}{y^2}[1 - \ln y]\ ; \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{2x}{y^2}[1 - \ln y]$$

22. $$\left.\begin{aligned} &\nabla\cdot\bar{i}_x = \nabla\cdot\bar{i}_y = \nabla\cdot\bar{i}_z = 0\\ &\nabla\times\bar{i}_x = \nabla\times\bar{i}_y = \nabla\times\bar{i}_z = 0\end{aligned}\right\}\ \text{Cartesian}$$

Cylindrical:

$$\nabla\cdot\overline{i}_r = \frac{1}{r}\frac{\partial r}{\partial r} = \frac{1}{r};\quad \nabla\cdot\overline{i}_\phi = 0;\quad \nabla\cdot\overline{i}_z = 0$$

$$\nabla\times\overline{i}_r = 0;\quad \nabla\times\overline{i}_\phi = \frac{1}{r}\frac{\partial r}{\partial r}\overline{i}_z = \frac{1}{r}\overline{i}_z;\quad \nabla\times\overline{i}_z = 0$$

Spherical:

$$\nabla\cdot\overline{i}_r = \frac{1}{r^2}\frac{\partial r^2}{\partial r} = \frac{2}{r};\quad \nabla\cdot\overline{i}_\theta = \frac{1}{r\sin\theta}\frac{\partial\sin\theta}{\partial\theta} = \frac{\cot\theta}{r};\quad \nabla\cdot\overline{i}_\phi = 0$$

$$\nabla\times\overline{i}_r = 0;\quad \nabla\times\overline{i}_\theta = \frac{1}{r}\frac{\partial r}{\partial r}\overline{i}_\phi = \frac{1}{r}\overline{i}_\phi;$$

$$\nabla\times\overline{i}_\phi = \frac{1}{r\sin\theta}\frac{\partial\sin\theta}{\partial\theta}\overline{i}_r - \frac{1}{r}\frac{\partial r}{\partial r}\overline{i}_\theta = \frac{\cot\theta}{r}\overline{i}_r - \frac{1}{r}\overline{i}_\theta$$

23.

	Cartesian	Cylindrical	Spherical
a)	$h_x = 1$	$h_r = 1$	$h_r = 1$
	$h_y = 1$	$h_\phi = r$	$h_\theta = r$
	$h_z = 1$	$h_z = 1$	$h_\phi = r\sin\theta$

b) $$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw$$

$$= \nabla f\cdot\overline{d\ell}$$

$$= \nabla f\cdot[h_u du\overline{i}_u + h_v dv\overline{i}_v + h_w dw\overline{i}_w]$$

$$(\nabla f)_u = \frac{1}{h_u}\frac{\partial f}{\partial u};\quad (\nabla f)_v = \frac{1}{h_v}\frac{\partial f}{\partial v};\quad (\nabla f)_w = \frac{1}{h_w}\frac{\partial f}{\partial w}$$

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\overline{i}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\overline{i}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\overline{i}_w$$

c) $$dS_u = h_v h_w\, dvdw;\quad dS_v = h_u h_w\, dudw;\quad dS_w = h_u h_v\, dudv$$

$$dV = h_u h_v h_w\, dudvdw$$

d) Divergence

$$\Phi = \oint_S \overline{A}\cdot\overline{dS} = \int_{\substack{1\\u}} A_u h_v h_w dvdw - \int_{\substack{1'\\u-\Delta u}} A_u h_v h_w dvdw$$

$$+ \int_{\substack{2\\v+\Delta v}} A_v h_u h_w dudw - \int_{\substack{2'\\v}} A_v h_u h_w dudw$$

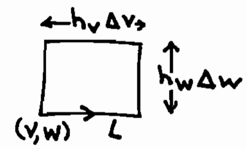
$$+ \int_{\substack{3\\w+\Delta w}} A_w h_u h_v dudv - \int_{\substack{3'\\w}} A_w h_u h_v dudv$$

$$= \left\{ \frac{A_u h_v h_w \big|_u - A_u h_v h_w \big|_{u-\Delta u}}{\Delta u} + \frac{A_v h_u h_w \big|_{v+\Delta v} - A_v h_u h_w \big|_v}{\Delta v} + \frac{A_w h_u h_v \big|_{w+\Delta w} - A_w h_u h_v \big|_w}{\Delta w} \right\} \Delta u \Delta v \Delta w$$

$$\nabla \cdot \overline{A} = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0 \\ \Delta w \to 0}} \frac{\oint_S \overline{A} \cdot \overline{dS}}{\Delta V} = \frac{\oint_S \overline{A} \cdot \overline{dS}}{h_u h_v h_w \Delta u \Delta v \Delta w}$$

$$= \frac{1}{h_u h_v h_w} \left[\frac{\partial (h_v h_w A_u)}{\partial u} + \frac{\partial (h_u h_w A_v)}{\partial v} + \frac{\partial (h_u h_v A_w)}{\partial w} \right]$$

<u>Curl</u>



$$(\nabla \times \overline{A})_u = \lim_{\substack{\Delta v \to 0 \\ \Delta w \to 0}} \frac{\oint_L \overline{A} \cdot \overline{d\ell}}{h_v h_w \Delta v \Delta w}$$

$$\oint_L \overline{A} \cdot \overline{d\ell} = \left[A_v h_v \Delta v \big|_w - A_v h_v \Delta v \big|_{w+\Delta w} \right] + \left[A_w h_w \Delta w \big|_{v+\Delta v} - A_w h_w \Delta w \big|_v \right]$$

$$(\nabla \times \overline{A})_u = \lim_{\substack{\Delta v \to 0 \\ \Delta w \to 0}} \frac{1}{h_v h_w} \left\{ \frac{\left[A_v h_v \big|_w - A_v h_v \big|_{w+\Delta w} \right]}{\Delta w} + \frac{\left[A_w h_w \big|_{v+\Delta v} - A_w h_w \big|_v \right]}{\Delta v} \right\}$$

$$= \frac{1}{h_v h_w} \left[\frac{\partial (h_w A_w)}{\partial v} - \frac{\partial (h_v A_v)}{\partial w} \right]$$

<u>Similarly</u>

$$(\nabla \times \overline{A})_v = \frac{1}{h_u h_w} \left[\frac{\partial (h_u A_u)}{\partial w} - \frac{\partial (h_w A_w)}{\partial u} \right]$$

$$(\nabla \times \overline{A})_w = \frac{1}{h_u h_v} \left[\frac{\partial (h_v A_v)}{\partial u} - \frac{\partial (h_u A_u)}{\partial v} \right]$$

e) $$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} \left(\frac{h_v h_w}{h_u} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_u h_w}{h_v} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_u h_v}{h_w} \frac{\partial f}{\partial w} \right) \right]$$

25. a) $\bar{r}_{QP} = (x_2 - x_1)\bar{i}_x + (y_2 - y_1)\bar{i}_y + (z_2 - z_1)\bar{i}_z$

$$r_{QP} = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$$

$$= [1^2 + (-5)^2 + 2^2]^{1/2}$$

$$= \sqrt{30}$$

b) $\bar{i}_{QP} = \dfrac{\bar{r}_{QP}}{r_{QP}} = \dfrac{\bar{i}_x - 5\bar{i}_y + 2\bar{i}_z}{\sqrt{30}}$

c) $\bar{n} = n_x\bar{i}_x + n_y\bar{i}_y$

$$\bar{n}\cdot\bar{i}_{QP} = 0 \rightarrow n_x - 5n_y = 0 \rightarrow \bar{n} = \frac{5\bar{i}_x + \bar{i}_y}{\sqrt{26}}$$

Miscellaneous

26. $L\dfrac{di}{dt} + \dfrac{1}{C}\int i\,dt + iR = V_o$

a) $\dfrac{d^2i}{dt^2} + \dfrac{R}{L}\dfrac{di}{dt} + \dfrac{i}{LC} = 0$

b) $i = \hat{I}e^{st}$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

c) Initial conditions

$i(t=0) = 0$

$L\dfrac{di}{dt}\Big|_{t=0} = V_o$

Steady state: $v_L = 0$, $v_c = V_o$, $v_R = 0$

d) $\alpha = \dfrac{R}{2L}$, $\beta = \sqrt{-\left(\dfrac{R}{2L}\right)^2 + \dfrac{1}{LC}}$

$$s = -\alpha \pm j\beta$$

$$i(t) = [A_1 \sin\beta t + A_2 \cos\beta t]e^{-\alpha t}$$

$$\frac{di}{dt} = -\alpha[A_1 \sin\beta t + A_2 \cos\beta t]e^{-\alpha t} + \beta[A_1 \cos\beta t - A_2 \sin\beta t]e^{-\alpha t}$$

$$i(t=0) = 0 \rightarrow A_2 = 0$$

$$\left.\frac{di}{dt}\right|_{t=0} = \frac{V_o}{L} = -\alpha \cancelto{0}{A_2} + \beta A_1 \rightarrow A_1 = \frac{V_o}{L\beta}$$

$$i(t) = \frac{V_o}{L\beta} \sin\beta t e^{-\alpha t}$$

If β is imaginary

$$i(t) = \frac{V_o}{L} \frac{\sinh|\beta|t}{|\beta|} e^{-\alpha t}$$

If $\beta = 0$

$$i(t) = \frac{V_o t}{L} e^{-\alpha t}$$

e) $V_R = iR = \dfrac{V_o R}{L\beta} \sin\beta t e^{-\alpha t}$

$$V_C = \frac{1}{C} \int i(t)dt = \frac{-V_o}{LC\beta} \frac{[\alpha\sin\beta t + \beta\cos\beta t]e^{-\alpha t}}{\alpha^2 + \beta^2}$$

$$= \frac{-V_o}{\beta} [\alpha\sin\beta t + \beta\cos\beta t]e^{-\alpha t} + \text{constant}$$

$$V_C(t=0) = 0 \rightarrow \text{constant} = V_o \rightarrow V_C = \frac{-V_o}{\beta} [\alpha\sin\beta t + \beta\cos\beta t]e^{-\alpha t} + V_o$$

$$V_L = \frac{Ldi}{dt} = \frac{V_o}{\beta} [\beta\cos\beta t - \alpha\sin\beta t]e^{-\alpha t}$$

f) $i(t) = [A_1 \sin\beta t + A_2 \cos\beta t]e^{-\alpha t}$

$$\left.\begin{array}{l} i(t=0) = 0 \\ L \left.\dfrac{di}{dt}\right|_{t=0} = -V_o \end{array}\right\} \rightarrow i(t) = \frac{-V_o}{L\beta} \sin\beta t e^{-\alpha t}$$

27. a) $Lj\omega I_n + \dfrac{(I_n - I_{n+1})}{Cj\omega} - \dfrac{(I_{n-1} - I_n)}{Cj\omega} = 0$

$$I_{n+1} - \left(2 - \frac{\omega^2}{\omega_o^2}\right) I_n + I_{n-1} = 0; \ \omega_o^2 = \frac{1}{LC}$$

b) $I_n = \hat{I}\lambda^n$

$$\hat{I}[\lambda^{n+1} - \left(2 - \frac{\omega^2}{\omega_o^2}\right)\lambda^n + \lambda^{n-1}] = 0$$

$$\hat{I}\lambda^{n-1}\underbrace{[\lambda^2 - (2 - \frac{\omega^2}{\omega_o^2})\lambda + 1]}_{0} = 0$$

$$\lambda = (1 - \frac{\omega^2}{2\omega_o^2}) \pm \sqrt{(1 - \frac{\omega^2}{2\omega_o^2})^2 - 1}$$

$$\lambda_1 = (1 - \frac{\omega^2}{2\omega_o^2}) + \sqrt{(1 - \frac{\omega^2}{2\omega_o^2})^2 - 1}$$

$$\lambda_2 = (1 - \frac{\omega^2}{2\omega_o^2}) - \sqrt{(1 - \frac{\omega^2}{2\omega_o^2})^2 - 1}$$

$$= \frac{1}{\lambda_1}$$

c) $I_n = A\lambda_1^n + B\lambda_1^{-n}$

$$V_n = \frac{I_n - I_{n+1}}{Cj\omega} = \frac{A\lambda_1^n(1 - \lambda_1) + B\lambda_1^{-n}(1 - \lambda_1^{-1})}{Cj\omega}$$

At source end $(n=0) \rightarrow I_o = A + B$

<u>Open Circuited End</u>

$$I_N = A_1\lambda_1^N + B\lambda_1^{-N} = 0 \rightarrow A = \frac{-I_o\lambda_1^{-2N}}{1 - \lambda_1^{-2N}}, \quad B = \frac{I_o}{1 - \lambda_1^{-2N}}$$

$$I_n = \frac{I_o}{1 - \lambda_1^{-2N}} [-\lambda_1^{n-2N} + \lambda_1^{-n}]$$

$$V_n = \frac{I_o}{Cj\omega(1 - \lambda_1^{-2N})} [-\lambda_1^{n-2N}(1 - \lambda_1) + \lambda_1^{-n}(1 - \lambda_1^{-1})]$$

<u>Short Circuited End</u>

$$V_N = \frac{A\lambda_1^N(1 - \lambda_1) + B\lambda_1^{-N}(1 - \lambda_1^{-1})}{Cj\omega} = 0$$

$$A = \frac{I_o \lambda_1^{-(N+1)}}{\lambda_1^{-(N+1)} + \lambda_1^N}, \quad B = \frac{I_o \lambda_1^N}{\lambda_1^{-(N+1)} + \lambda_1^N}$$

$$I_n = \frac{I_o}{\lambda_1^{-(N+1)} + \lambda_1^N} [\lambda_1^{n-N-1} + \lambda_1^{-n+N}]$$

$$V_n = \frac{I_o(1 - \lambda_1)}{Cj\omega(\lambda_1^{-(N+1)} + \lambda_1^N)} [\lambda_1^{n-N-1} - \lambda_1^{-n+N-1}]$$

d) If $I_o = 0$, for an open circuited end, I_n and V_n can be non-zero if

$$\lambda_1^{-2N} = 1 \rightarrow \lambda_1 = (1)^{\frac{1}{2N}} = e^{j2\pi r/2N} \qquad r = 1, 2, \ldots 2N$$

$$\left(1 - \frac{\omega^2}{2\omega_o^2}\right) \pm \sqrt{\left(1 - \frac{\omega^2}{2\omega_o^2}\right)^2 - 1} = e^{j2\pi r/2N}$$

$$\pm\sqrt{\left(1 - \frac{\omega^2}{2\omega_o^2}\right)^2 - 1} = e^{j2\pi r/2N} - \left(1 - \frac{\omega^2}{2\omega_o^2}\right)$$

Square both sides and reduce

$$\left(1 - \frac{\omega^2}{2\omega_o^2}\right)^2 - 1 = e^{j4\pi r/2N} + \left(1 - \frac{\omega^2}{2\omega_o^2}\right)^2 - 2\left(1 - \frac{\omega^2}{2\omega_o^2}\right)e^{j2\pi r/2N}$$

$$\left(1 - \frac{\omega^2}{2\omega_o^2}\right) = \frac{e^{j2\pi r/2N} + e^{-j2\pi r/2N}}{2} = \cos\frac{\pi r}{N}$$

$$\omega^2 = 2\omega_o^2 \left(1 - \cos\frac{\pi r}{N}\right); \quad r = 1, 2, \ldots 2N$$

For a short circuited end

$$\lambda_1^N + \lambda_1^{-(N+1)} = 0 \rightarrow \lambda_1^{2N+1} = -1 \rightarrow \lambda_1 = (-1)^{1/(2N+1)} = e^{j\pi(2r-1)/(2N+1)}$$

$$r = 1, 2, \ldots 2N+1$$

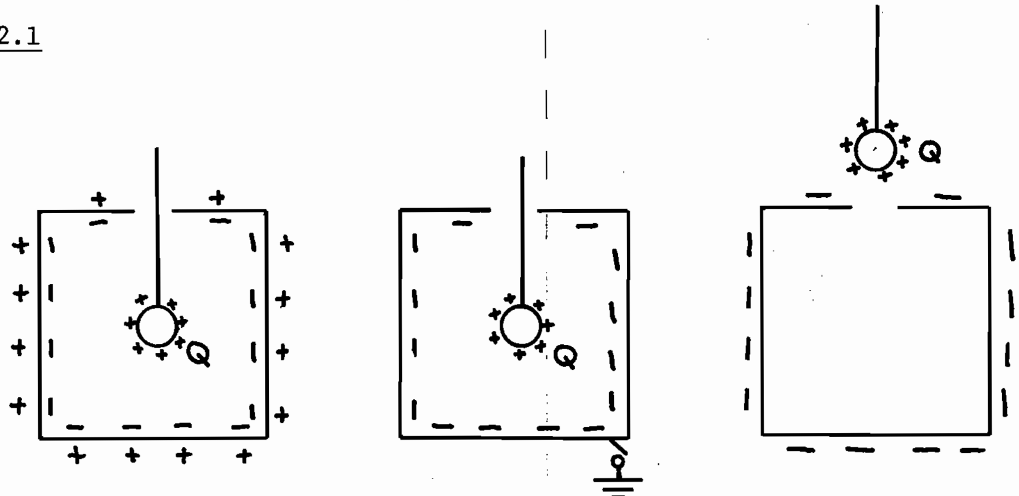
$$\rightarrow \omega^2 = 2\omega_o^2\left(1 - \cos\frac{\pi(2r-1)}{2N+1}\right)$$

CHAPTER 2
THE ELECTRIC FIELD

Section 2.1

1.

a)



b) Net charge on pail after charged ball is removed = $-Q$.

2. a) $\frac{Q}{2}$

b) $q_1 = \frac{Q}{2};\ q_2 = \frac{Q}{4};\ q_3 = \frac{Q}{8};\ . \ . \ . \ q_N = \frac{Q}{2^N}$

Original sphere has charge $q_N = \frac{Q}{2^N}$

c) $Q_T = Q[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + . \ . \ . \ \frac{1}{2^N} + \frac{1}{2^N}]$

$\frac{Q_T}{2} = Q[\quad \frac{1}{4} + \frac{1}{8} + . \ . \ . \ \frac{1}{2^N} + \frac{2}{2^{N+1}}]$

$\frac{Q_T}{2} = Q[\frac{1}{2} + \underbrace{\frac{1}{2^N} - \frac{2}{2^{N+1}}}_{0}]$ (Subtract)

$= \frac{Q}{2} \rightarrow Q_T = Q$

The total charge in the system remains unchanged.

3. $q\ E_o = \rho_m g\ \frac{4}{3}\ \pi R^3 \rightarrow E_o = \frac{4}{3}\ \pi R^3\ \frac{\rho_m g}{q}$

4. $\frac{Q_1 Q_2}{4\pi\varepsilon_o d^2} = T\sin\theta = \frac{Td}{2\ell}$ where T is the tension in the strings.

$$T = \frac{Mg}{\cos\theta} = \frac{Mg\ell}{\sqrt{\ell^2 - (\frac{d}{2})^2}}$$

$$Q_2 = \frac{2\pi\varepsilon_o d^3}{Q_1 \ell} T = \frac{2\pi\varepsilon_o d^3 Mg}{Q_1\sqrt{\ell^2 - (\frac{d}{2})^2}}$$

5. a) $m\omega^2 R = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2} \rightarrow \omega = \left[\frac{Q_1 Q_2}{4\pi\varepsilon_o R^3 m}\right]^{1/2}$

b) $Q_1 = e, \quad Q_2 = Ze$

$$L = mvR = m\omega R^2 = nh/2\pi$$

$$m^2\omega^2 R^4 = (\frac{nh}{2\pi})^2 = \frac{m^2 R^4 \; Q_1 Q_2}{4\pi\varepsilon_o R^3 m} = \frac{m\; Q_1 Q_2 \; R}{4\pi\varepsilon_o}$$

$$R = \frac{4\pi\varepsilon_o (\frac{nh}{2\pi})^2}{m\; Q_1 Q_2} = \frac{4\pi\varepsilon_o (\frac{nh}{2\pi})^2}{m\; Ze^2}$$

c) For $Z = 1, \; n = 1$

$$R = \frac{\frac{4\pi}{36\pi} \times 10^{-9} \; (\frac{h}{2\pi})^2}{9.1 \times 10^{-31} (1.6 \times 10^{-19})^2} = 5.3 \times 10^{-11} \text{ meters} \approx .53 \text{ A}^o$$

$$\omega = \left[\frac{e^2}{4\pi\varepsilon_o R^3 m}\right]^{1/2} = \left[\frac{(1.6 \times 10^{-19})^2}{\frac{10^{-9}}{9} (5.3 \times 10^{-11})^3 (9.1 \times 10^{-31})}\right]^{1/2} = 4.1 \times 10^{16} \text{ rad/sec}$$

$$v = \omega R = 2.2 \times 10^6 \text{ m/sec}$$

6. $T = \frac{Mg}{\cos\theta} = \frac{Q_1 Q_2}{4\pi\varepsilon_o s^2 \sin\theta}$ where $\sin\theta = \frac{s}{2\ell}$ $(Q_1 = Q_2 = \frac{Q}{2})$

$$\frac{Mg 4\pi\varepsilon_o (2\ell\sin\theta)^2 \sin\theta}{Q_1 Q_2 \cos\theta} = 1$$

$$\sin^2\theta \tan\theta = \frac{Q_1 Q_2}{16\pi\varepsilon_o \ell^2 Mg} = \frac{Q^2}{64\pi\varepsilon_o \ell^2 Mg}$$

7. a) $$m_1 \frac{d^2 r_1}{dt^2} = -\frac{q_1 q_2}{4\pi\varepsilon_o (r_2 - r_1)^2} \qquad \text{(multiply by } m_2\text{)}$$

$$m_2 \frac{d^2 r_2}{dt^2} = \frac{q_1 q_2}{4\pi\varepsilon_o (r_2 - r_1)^2} \qquad \text{(multiply by } m_1\text{)}$$

$$m_1 m_2 \frac{d^2 (r_2 - r_1)}{dt^2} = \frac{q_1 q_2 \, (m_2 + m_1)}{4\pi\varepsilon_o (r_2 - r_1)^2} \qquad \text{(subtract)}$$

$$m \frac{d^2 r}{dt^2} = \frac{q_1 q_2}{4\pi\varepsilon_o r^2} \ ; \qquad m = \frac{m_1 m_2}{m_1 + m_2} \ , \qquad r = r_2 - r_1$$

b) $$m \frac{dv}{dt} = m \frac{dv}{dr}\frac{dr}{dt} = mv \frac{dv}{dr} = \frac{q_1 q_2}{4\pi\varepsilon_o r^2}$$

$$vdv = \frac{q_1 q_2}{4\pi\varepsilon_o m} \frac{dr}{r^2} \rightarrow \frac{1}{2} v^2 = -\frac{q_1 q_2}{4\pi\varepsilon_o m}\left(\frac{1}{r} - \frac{1}{r_o}\right)$$

$$v = \pm\sqrt{\frac{-q_1 q_2}{2\pi\varepsilon_o m}\left(\frac{1}{r} - \frac{1}{r_o}\right)} \qquad \begin{array}{l} q_1 q_2 > 0, \quad r > r_o, \quad v > 0 \\ q_1 q_2 < 0, \quad r < r_o, \quad v < 0 \end{array}$$

c) $$v = \frac{dr}{dt} = \pm\sqrt{\frac{-q_1 q_2}{2\pi\varepsilon_o m}\left(\frac{1}{r} - \frac{1}{r_o}\right)}$$

$$\rightarrow \frac{dr}{\left[\frac{1}{r} - \frac{1}{r_o}\right]^{1/2}} = \frac{\sqrt{r}\ dr}{\left[1 - \frac{r}{r_o}\right]^{1/2}} = \pm\left[\frac{-q_1 q_2}{2\pi m\varepsilon_o}\right]^{1/2} dt$$

i) $q_1 q_2 < 0$ (charges attract)

$$\sqrt{r_o}\left[-\sqrt{r}\ \sqrt{r_o - r} + r_o \sin^{-1}\sqrt{\frac{r}{r_o}}\right] = -\left[\frac{-q_1 q_2}{2\pi m\varepsilon_o}\right]^{1/2} t + \text{constant}$$

$$\sqrt{r_o}\left[-\sqrt{r}\ \sqrt{r_o - r} + r_o\left[\sin^{-1}\sqrt{\frac{r}{r_o}} - \frac{\pi}{2}\right]\right] = -\left[\frac{-q_1 q_2}{2\pi m\varepsilon_o}\right]^{1/2} t$$

$$r(t{=}0) = r_o \rightarrow \text{constant} = \frac{\pi r_o^{3/2}}{2}$$

ii) $q_1q_2 > 0$ (charges repel)

$$\sqrt{r_o}\left[\sqrt{r}\,\sqrt{r-r_o} + r_o \ln\left[\sqrt{r} + \sqrt{r-r_o}\right]\right] = \left[\frac{q_1q_2}{2\pi m\varepsilon_o}\right]^{1/2} t + \text{constant}$$

$$\sqrt{r_o}\left[\sqrt{r}\,\sqrt{r-r_o} + r_o \ln\left[\sqrt{\frac{r}{r_o}} + \sqrt{\frac{r}{r_o} - 1}\right]\right] = \left[\frac{q_1q_2}{2\pi m\varepsilon_o}\right]^{1/2} t$$

$$r(t=0) = r_o \rightarrow \text{constant} = r_o^{3/2} \ln\sqrt{r_o}$$

d) If $q_1q_2 < 0$ see (c)(i)

$r = 0$ when

$$\left[\frac{-q_1q_2}{2\pi m\varepsilon_o}\right]^{1/2} t = \frac{\pi}{2} r_o^{3/2} \rightarrow t = \frac{\pi}{2} r_o^{3/2} \left[\frac{2\pi m\varepsilon_o}{-q_1q_2}\right]^{1/2}$$

e) $$\beta_1 \frac{dr_1}{dt} = -\frac{q_1q_2}{4\pi\varepsilon_o(r_2 - r_1)^2} \qquad \text{(multiply by } \beta_2)$$

$$\beta_2 \frac{dr_2}{dt} = \frac{q_1q_2}{4\pi\varepsilon_o(r_2 - r_1)^2} \qquad \text{(multiply by } \beta_1)$$

$$\beta_2\beta_1 \frac{d}{dt}(r_2 - r_1) = \frac{q_1q_2}{4\pi\varepsilon_o(r_2 - r_1)^2}(\beta_2 + \beta_1) \qquad \text{(subtract)}$$

$$\beta \frac{dr}{dt} = \frac{q_1q_2}{4\pi\varepsilon_o r^2} \; ; \quad \beta = \frac{\beta_1\beta_2}{\beta_1 + \beta_2} \, , \quad r = r_2 - r_1$$

$$r^2 dr = \frac{q_1q_2}{4\pi\varepsilon_o\beta} dt$$

$$\frac{r^3}{3} = \frac{q_1q_2}{4\pi\varepsilon_o\beta} t + \frac{r_o^3}{3}$$

If $q_1q_2 < 0$, $r = 0$ when

$$\frac{-q_1q_2 t}{4\pi\varepsilon_o\beta} = \frac{r_o^3}{3} \rightarrow t = -\frac{4\pi\varepsilon_o\beta r_o^3}{3q_1q_2}$$

8. $$m\frac{d^2z}{dt^2} = qE_o \rightarrow z = \frac{qE_o t^2}{2m} + \cancel{v_{zo}}^{\,0}t + \cancel{z_o}^{\,0}$$

$$m\frac{d^2x}{dt^2} = 0 \rightarrow x = v_o t \rightarrow t = \frac{x}{v_o}$$

$$z = \frac{qE_o}{2m}\frac{x^2}{v_o^2}$$

$$z(x{=}L) = h = \frac{qE_o L^2}{2mv_o^2}$$

9. $$m\frac{d^2\xi}{dt^2} = -mg\underbrace{\sin\theta}_{\approx \frac{\xi}{\ell}} + \frac{qQ}{4\pi\varepsilon_o}\left[\frac{1}{(D+\xi)^2} - \frac{1}{(D-\xi)^2}\right]$$

$$\approx \frac{-mg\xi}{\ell} + \frac{qQ}{4\pi\varepsilon_o D^2}\left\{\frac{1}{(1+\frac{\xi}{D})^2} - \frac{1}{(1-\frac{\xi}{D})^2}\right\}$$

$$\approx \frac{-mg\xi}{\ell} + \frac{qQ}{4\pi\varepsilon_o D^2}\left\{1 - \frac{2\xi}{D} - (1 + \frac{2\xi}{D})\right\}$$

$$= \frac{-mg\xi}{\ell} - \frac{qQ}{\pi\varepsilon_o D^3}\xi$$

$$m\frac{d^2\xi}{dt^2} = -(\frac{mg}{\ell} + \frac{qQ}{\pi\varepsilon_o D^3})\,\xi$$

$$\xi = A_1\sin\omega_o t + A_2\cos\omega_o t; \quad \omega_o^2 = \frac{g}{\ell} + \frac{qQ}{m\pi\varepsilon_o D^3}$$

Unstable when

$$\omega_o^2 < 0 \rightarrow qQ < \frac{-mg}{\ell}\pi\varepsilon_o D^3$$

If $\xi(t{=}0) = \xi_o$, $\frac{\partial\xi}{\partial t}(t{=}0) = 0$

$$\xi = \xi_o \cos\omega_o t$$

10. a) $$\overline{F}_q = \frac{qQ\sqrt{3}}{4\pi\varepsilon_o a^2}\,\overline{i}_y$$

b) $$\overline{E}_p = \frac{1}{4\pi\varepsilon_o}\left[\frac{2Q}{(\frac{7}{16})^{3/2}a^2}\frac{\sqrt{3}}{4} - \frac{q}{\frac{3}{16}a^2}\right]\overline{i}_y$$

$$= \frac{4}{\pi\varepsilon_o a^2}\left[\frac{2\sqrt{3}\,Q}{(7)^{3/2}} - \frac{q}{3}\right]\overline{i}_y$$

$$\overline{E}_p = 0 \rightarrow q = \frac{6\sqrt{3}}{7^{3/2}}\,Q$$

11. $$\overline{E}_1 = \frac{q_1\,[-\frac{a}{2}\,\overline{i}_x + \frac{a}{2}\,\overline{i}_y + z\overline{i}_z]}{4\pi\varepsilon_o[z^2 + \frac{a^2}{2}]^{3/2}}$$

$$\overline{E}_2 = \frac{q_2\,[\frac{a}{2}\,\overline{i}_x + \frac{a}{2}\,\overline{i}_y + z\overline{i}_z]}{4\pi\varepsilon_o[z^2 + \frac{a^2}{2}]^{3/2}}$$

$$\overline{E}_3 = \frac{q_3\,[\frac{a}{2}\,\overline{i}_x - \frac{a}{2}\,\overline{i}_y + z\overline{i}_z]}{4\pi\varepsilon_o[z^2 + \frac{a^2}{2}]^{3/2}}$$

$$\overline{E}_4 = \frac{q_4\,[-\frac{a}{2}\,\overline{i}_x - \frac{a}{2}\,\overline{i}_y + z\overline{i}_z]}{4\pi\varepsilon_o[z^2 + \frac{a^2}{2}]^{3/2}}$$

$$\overline{E}_T(z) = \overline{E}_1 + \overline{E}_2 + \overline{E}_3 + \overline{E}_4$$

a) $$q_1 = q_2 = q_3 = q_4 \equiv q \rightarrow \overline{E}_T(z) = \frac{4qz}{4\pi\varepsilon_o(z^2 + \frac{a^2}{2})^{3/2}}\,\overline{i}_z$$

b) $$q_1 = q_3 \equiv q,\; q_2 = q_4 \equiv -q \rightarrow \overline{E}_T(z) = 0$$

c) $$q_1 = q_2 \equiv q,\; q_3 = q_4 \equiv -q \rightarrow \overline{E}_T(z) = \frac{2qa}{4\pi\varepsilon_o(z^2 + \frac{a^2}{2})^{3/2}}\,\overline{i}_y$$

12. a)
$$q = \int_{-\infty}^{0} \lambda_o e^{z/a} dz + \int_{0}^{\infty} \lambda_o e^{-z/a} dz$$

$$= \lambda_o a\ e^{z/a}\Big|_{-\infty}^{0} - \lambda_o a\ e^{-z/a}\Big|_{0}^{\infty}$$

$$= 2\lambda_o a$$

b)
$$q = 4\pi \int_0^\infty \frac{\rho_o\ r^2 dr}{[1 + \frac{r}{a}]^4}$$

Let $u = 1 + \frac{r}{a}$; $du = \frac{1}{a} dr$

$$q = 4\pi \int_1^\infty \frac{\rho_o a^3 (u - 1)^2}{u^4} du$$

$$= 4\pi\rho_o a^3 \int_1^\infty \frac{u^2 - 2u + 1}{u^4} du$$

$$= 4\pi\rho_o a^3 \int_1^\infty \left[\frac{1}{u^2} - \frac{2}{u^3} + \frac{1}{u^4}\right] du$$

$$= 4\pi\rho_o a^3 \left[-\frac{1}{u} + \frac{1}{u^2} - \frac{1}{3u^3}\right]\Big|_1^\infty$$

$$= \frac{4}{3}\pi\rho_o a^3$$

c)
$$q = \int_{y=-\infty}^{+\infty} \int_{x=0}^{\infty} \frac{2\sigma_o e^{-x/a}}{[1 + (\frac{y}{b})^2]} dxdy$$

$$= \int_{y=-\infty}^{+\infty} \frac{-2\sigma_o a e^{-x/a}\Big|_0^\infty}{[1 + (\frac{y}{b})^2]} dy$$

$$= 2\sigma_o a \int_{-\infty}^{+\infty} \frac{dy}{[1 + (\frac{y}{b})^2]}$$

$$= 2\sigma_o ab \tan^{-1}\frac{y}{b}\Big|_{-\infty}^{+\infty}$$

$$= 2\sigma_o ab\pi$$

13. a) $M \dfrac{d^2x}{dt^2} = \dfrac{q\sigma_o}{2\varepsilon_o} - Mg$

$$x = \left(\frac{q\sigma_o}{2\varepsilon_o M} - g\right)\frac{t^2}{2} + \overset{0}{\cancel{v_o}}\,t + x_o$$

b) Charge remains stationary if

$$\frac{q\sigma_o}{2M\varepsilon_o} - g = 0 \rightarrow \sigma_o = \frac{2Mg\varepsilon_o}{q}$$

c) $x = 0$ when $\left(\dfrac{q\sigma_o}{2M\varepsilon_o} - g\right)\dfrac{t^2}{2} = -x_o$

$$t = \left[\frac{2x_o}{g - \dfrac{q\sigma_o}{2M\varepsilon_o}}\right]^{1/2}$$

$$v = \frac{dx}{dt} = \left(\frac{q\sigma_o}{2M\varepsilon_o} - g\right)t = -\left[2x_o\left[g - \frac{q\sigma_o}{2M\varepsilon_o}\right]\right]^{1/2}$$

14. a) $\overline{F}_q = \dfrac{q\lambda_o}{2\pi\varepsilon_o D}\,\overline{i}_y$

b) $$\overline{F}_\ell = -\int_{-\infty}^{+\infty} \frac{\lambda_o dz\, q\, D}{4\pi\varepsilon_o (D^2 + z^2)^{3/2}}\,\overline{i}_y$$

$$= \frac{-q\lambda_o D}{4\pi\varepsilon_o}\int_{-\infty}^{+\infty} \frac{dz}{(D^2 + z^2)^{3/2}}\,\overline{i}_y$$

$$= \frac{-q\lambda_o D}{4\pi\varepsilon_o}\,\frac{z}{D^2\sqrt{D^2 + z^2}}\,\Bigg|_{-\infty}^{+\infty}\,\overline{i}_y$$

$$= \frac{-q\lambda_o}{2\pi\varepsilon_o D}\,\overline{i}_y$$

c) $$\overline{F}_q = -\overline{F}_\ell = \frac{q}{4\pi\varepsilon_o}\int_0^{\infty} \frac{2\lambda_o zD\, dz}{a(D^2 + z^2)^{3/2}}\,\overline{i}_y$$

$$= \frac{\lambda_o Dq}{2\pi\varepsilon_o a}\int_0^{\infty} \frac{zdz}{(D^2 + z^2)^{3/2}}\,\overline{i}_y$$

$$= \frac{\lambda_o Dq}{2\pi\varepsilon_o a}\,\frac{-1}{\sqrt{D^2 + z^2}}\,\Bigg|_0^{\infty}\,\overline{i}_y$$

$$= \frac{\lambda_o q}{2\pi\varepsilon_o a} \bar{i}_y$$

15. $T = \frac{Mg}{\cos\theta} = \frac{Q\sigma_o}{2\varepsilon_o \sin\theta} \rightarrow \tan\theta = \frac{Q\sigma_o}{2\varepsilon_o Mg}$

16. a) $dE_r = \frac{\lambda\, dz\, r}{4\pi\varepsilon_o (r^2 + z^2)^{3/2}}$

$$E_r = \frac{\lambda r}{4\pi\varepsilon_o} \int_{-L}^{L} \frac{dz}{(r^2 + z^2)^{3/2}}$$

$$= \frac{\lambda r}{4\pi\varepsilon_o} \frac{z}{r^2 \sqrt{z^2 + r^2}} \Bigg|_{-L}^{+L}$$

$$= \frac{\lambda}{2\pi\varepsilon_o r} \frac{L}{\sqrt{L^2 + r^2}}$$

b) $dE_x = \frac{\sigma_o\, dy}{2\pi\varepsilon_o \sqrt{x^2 + y^2}} \frac{L\, x}{\sqrt{L^2 + x^2 + y^2}\sqrt{x^2 + y^2}}$; $y > o$

$$= \frac{\sigma_o\, Lxdy}{2\pi\varepsilon_o (x^2 + y^2)\sqrt{L^2 + x^2 + y^2}}$$

$$E_x = \frac{\sigma_o Lx}{2\pi\varepsilon_o} \int_{-\infty}^{+\infty} \frac{dy}{(x^2 + y^2)\sqrt{L^2 + x^2 + y^2}}$$

Let $u = x^2 + y^2$, $y = \sqrt{u - x^2}$; $du = 2ydy = 2\sqrt{u - x^2}\, dy$

$$E_x = \frac{\sigma_o Lx}{2\pi\varepsilon_o} \int_{x^2}^{\infty} \frac{du}{u\sqrt{u - x^2}\sqrt{L^2 + u}}$$

$$= \frac{\sigma_o Lx}{2\pi\varepsilon_o} \int_{x^2}^{\infty} \frac{du}{u\sqrt{u^2 + u(L^2 - x^2) - L^2 y^2}}$$

$$= \frac{\sigma_o Lx}{2\pi\varepsilon_o} \frac{1}{Lx} \sin^{-1} \frac{(L^2 - x^2)u - 2L^2 x^2}{u\sqrt{(L^2 - x^2)^2 + 4L^2 x^2}} \Bigg|_{u=x^2}^{\infty}$$

$$= \frac{\sigma_o}{2\pi\varepsilon_o} \sin^{-1} \frac{(L^2 - x^2)u - 2L^2 x^2}{u(L^2 + x^2)} \Bigg|_{u=x^2}^{\infty}$$

$$= \frac{\sigma_o}{2\pi\varepsilon_o}\left[\sin^{-1}\frac{(L^2 - x^2)}{L^2 + x^2} + \frac{\pi}{2}\right]$$

$$\lim_{L\to\infty} E_x = \frac{\sigma_o}{2\varepsilon_o}$$

17. a) $$dE_y = \frac{-\sigma_o R d\phi}{2\pi\varepsilon_o R}\sin\phi$$

$$E_y = \frac{-\sigma_o}{2\pi\varepsilon_o}\int_0^\pi \sin\phi d\phi$$

$$= \frac{+\sigma_o}{2\pi\varepsilon_o}\cos\phi\Big|_0^\pi$$

$$= \frac{-\sigma_o}{\pi\varepsilon_o}$$

b) $$dE_y = \frac{-\rho_o dr}{\pi\varepsilon_o}$$

$$E_y = \frac{-\rho_o}{\pi\varepsilon_o}\int_{r=0}^{R} dr$$

$$= \frac{-\rho_o}{\pi\varepsilon_o} r\Big|_{r=0}^{R}$$

$$= \frac{-\rho_o R}{\pi\varepsilon_o}$$

c) <u>Surface charged hemi-sphere</u>

$$dE_z = \frac{-\sigma_o}{4\pi\varepsilon_o}\sin\theta\cos\theta\, d\theta d\phi$$

$$E_z = \frac{-\sigma_o}{4\pi\varepsilon_o}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\, d\theta d\phi$$

$$= \frac{-\sigma_o}{2\varepsilon_o}\int_{\theta=0}^{\pi/2}\frac{\sin 2\theta}{2}\, d\theta$$

$$= \frac{+\sigma_o}{8\varepsilon_o} \cos 2\theta \Big|_0^{\pi/2}$$

$$= \frac{-\sigma_o}{4\varepsilon_o}$$

Volume charged hemi-sphere of radius R

$$dE_z = \frac{-\rho_o dr}{4\varepsilon_o}$$

$$E_z = \frac{-\rho_o}{4\varepsilon_o} \int_0^R dr = \frac{-\rho_o R}{4\varepsilon_o}$$

18. a) $$\bar{E} = \frac{\lambda_o a}{4\pi\varepsilon_o [z^2+a^2]} \left\{ \int_o^{\pi} \frac{(z\bar{i}_z - a\bar{i}_r) d\phi}{\sqrt{z^2+a^2}} - \int_{\pi}^{2\pi} \frac{(z\bar{i}_z - a\bar{i}_r) d\phi}{\sqrt{z^2+a^2}} \right\}$$

$$= \frac{-\lambda_o a^2}{4\pi\varepsilon_o [z^2+a^2]^{3/2}} \left\{ \int_o^{\pi} [\cos\phi\,\bar{i}_x + \sin\phi\,\bar{i}_y] d\phi - \int_{\pi}^{2\pi} [\cos\phi\,\bar{i}_x + \sin\phi\,\bar{i}_y d\phi] \right\}$$

$$\bar{E} = \frac{-\lambda_o a^2}{\pi\varepsilon_o [z^2+a^2]^{3/2}} \bar{i}_y$$

b) $$dE_y = \frac{-\sigma_o r^2 dr}{\pi\varepsilon_o [z^2+r^2]^{3/2}} \; ; \quad E_y = \frac{-\sigma_o}{\pi\varepsilon_o} \int_o^a \frac{r^2 dr}{[z^2+r^2]^{3/2}}$$

$$= \frac{-\sigma_o}{\pi\varepsilon_o} \left[\frac{-r}{\sqrt{z^2+r^2}} + \ln(r + \sqrt{z^2+r^2}) \right] \Big|_o^a$$

$$= \frac{-\sigma_o}{\pi\varepsilon_o} \left[\frac{-a}{\sqrt{z^2+a^2}} + \ln\left(\frac{a+\sqrt{z^2+a^2}}{|z|}\right) \right]$$

19. a)

$$dE_z = \frac{\lambda_o dxz}{\pi\varepsilon_o (x^2 + \frac{a^2}{4} + z^2)^{3/2}}$$

$$E_z = \frac{\lambda_o z}{\pi\varepsilon_o} \int_{-a/2}^{+a/2} \frac{dx}{(x^2 + \frac{a^2}{4} + z^2)^{3/2}}$$

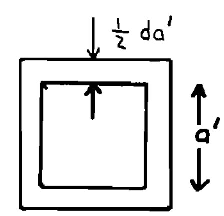
$$= \frac{\lambda_o z}{\pi\varepsilon_o} \frac{x}{(\frac{a^2}{4} + z^2)\sqrt{x^2 + \frac{a^2}{4} + z^2}} \Bigg|_{x=-a/2}^{+a/2}$$

$$= \frac{\lambda_o za}{\pi\varepsilon_o (\frac{a^2}{4} + z^2)\sqrt{\frac{a^2}{2} + z^2}}$$

Check

$$\lim_{z>>a} E_z = \pm \frac{\lambda_o a}{\pi\varepsilon_o z^2} = \pm \frac{Q_T}{4\pi\varepsilon_o z^2} \quad \begin{matrix} z > 0 \\ z < 0 \end{matrix} \quad (Q_T = 4\lambda_o a) \longleftarrow \text{field of point charge}$$

b) $$E_z = \int_0^a \frac{\sigma_o z\, a' \frac{da'}{2}}{\pi\varepsilon_o (\frac{a'^2}{4} + z^2)\sqrt{\frac{a'^2}{2} + z^2}}$$



$$u = z^2 + \frac{a'^2}{4} \; ; \quad du = \frac{a'\, da'}{2}$$

$$dE_z = \frac{\sigma_o z\, du}{\pi\varepsilon_o u\sqrt{2u - z^2}}$$

$$E_z = \frac{\sigma_o z}{\pi\varepsilon_o} \int_{z^2}^{z^2 + a^2/4} \frac{du}{u\sqrt{2u - z^2}}$$

$$= \frac{\sigma_o z}{\pi\varepsilon_o} \frac{2}{|z|} \tan^{-1} \sqrt{\frac{2u - z^2}{z^2}} \Bigg|_{u=z^2}^{z^2 + a^2/4}$$

$$= \pm \frac{2\sigma_o}{\pi\varepsilon_o}\left[\tan^{-1}\sqrt{\frac{z^2 + a^2/2}{z^2}} - \underbrace{\tan^{-1}1}_{\pi/4}\right] \begin{matrix} z > 0 \\ z < 0 \end{matrix}$$

$$\lim_{a\to\infty} E_z = \pm \frac{2\sigma_o}{\pi\varepsilon_o}\frac{\pi}{4} = \pm \frac{\sigma_o}{2\varepsilon_o} \quad \begin{matrix} z > 0 \\ z < 0 \end{matrix} \longleftarrow \text{field of sheet of surface charge}$$

20. a) $$dE_y = \frac{-2\lambda_o a^2 \sin\phi \, d\phi}{4\pi\varepsilon_o (a^2 + z^2)^{3/2}}$$

$$E_y = \frac{-2\lambda_o a^2}{4\pi\varepsilon_o (a^2 + z^2)^{3/2}} \int_0^{\pi} \sin\phi d\phi = \frac{-\lambda_o a^2}{\pi\varepsilon_o (a^2 + z^2)^{3/2}}$$

b) $$dE_y = \frac{-\sigma_o r^2 dr}{\pi\varepsilon_o (r^2 + z^2)^{3/2}}$$

$$E_y = \frac{-\sigma_o}{\pi\varepsilon_o}\int_0^a \frac{r^2 dr}{(r^2 + z^2)^{3/2}} = \frac{-\sigma_o}{\pi\varepsilon_o}\left[\frac{-r}{\sqrt{r^2 + z^2}} + \ln[r + \sqrt{r^2 + z^2}]\right]\Bigg|_{r=0}^{a}$$

$$= \frac{\sigma_o}{\pi\varepsilon_o}\left\{\frac{a}{\sqrt{a^2 + z^2}} - \ln\left[\frac{a + \sqrt{a^2 + z^2}}{|z|}\right]\right\}$$

c) $$E_y = \frac{-\lambda_o a^2}{4\pi\varepsilon_o (a^2 + z^2)^{3/2}} \underbrace{\int_0^{2\pi} \sin^2\phi d\phi}_{\pi}$$

$$= \frac{-\lambda_o a^2}{4\varepsilon_o (a^2 + z^2)^{3/2}}$$

d) $$E_y = \frac{\sigma_o}{4\varepsilon_o}\left\{\frac{a}{\sqrt{a^2 + z^2}} - \ln\left[\frac{a + \sqrt{a^2 + z^2}}{|z|}\right]\right\}$$

21. $$\overline{dE} = \frac{2\lambda_o (z\bar{i}_z - y\bar{i}_y)}{4\pi\varepsilon_o [a^2 + y^2 + z^2]^{3/2}} dy + \frac{\lambda_o a \, d\phi[-a\bar{i}_r + z\bar{i}_z]}{4\pi\varepsilon_o (z^2 + a^2)^{3/2}}$$

$$E_z = \frac{\lambda_o z}{2\pi\varepsilon_o}\int_{-\infty}^{0} \frac{dy}{[a^2 + y^2 + z^2]^{3/2}} + \frac{\lambda_o a z}{4\pi\varepsilon_o (z^2 + a^2)^{3/2}}\int_0^{\pi} d\phi$$

$$= \frac{\lambda_o z}{2\pi\varepsilon_o} \frac{y}{(a^2 + z^2)(a^2 + y^2 + z^2)^{1/2}} \Bigg|_{y=-\infty}^{0} + \frac{\lambda_o az}{4\varepsilon_o (z^2 + a^2)^{3/2}}$$

$$= \frac{\lambda_o z}{2\pi\varepsilon_o (a^2 + z^2)} \left[1 + \frac{\pi a}{2(a^2 + z^2)^{1/2}} \right]$$

$$E_y = \frac{-\lambda_o}{2\pi\varepsilon_o} \int_{-\infty}^{0} \frac{ydy}{[a^2 + y^2 + z^2]^{3/2}} - \frac{-\lambda_o a^2}{4\pi\varepsilon_o (z^2 + a^2)^{3/2}} \int_0^{\pi} \sin\phi d\phi$$

$$= \frac{+\lambda_o}{2\pi\varepsilon_o} \frac{1}{[a^2 + y^2 + z^2]^{1/2}} \Bigg|_{y=-\infty}^{0} + \frac{\lambda_o a^2}{4\pi\varepsilon_o (z^2 + a^2)^{3/2}} \cos\phi \Bigg|_0^{\pi}$$

$$= \frac{\lambda_o}{2\pi\varepsilon_o (a^2 + z^2)^{1/2}} \left[1 - \frac{a^2}{(z^2 + a^2)} \right] = \frac{\lambda_o z^2}{2\pi\varepsilon_o (a^2 + z^2)^{3/2}}$$

Section 2.4

22. a) $\overline{E} = ar^2 \overline{i}_r \rightarrow \nabla \cdot \overline{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) = 4Ar = \frac{\rho}{\varepsilon_o}$

$$Q_T = \int_V \rho \, dV$$

$$= \int_V 4\varepsilon_o A r^3 \sin\theta dr d\theta d\phi$$

$$= 16\pi\varepsilon_o A \int_0^R r^3 dr$$

$$= 4\pi\varepsilon_o AR^4$$

Another Method

$$Q_T = \int_S \varepsilon_o \overline{E} \cdot \overline{dS} = \varepsilon_o E_r (r=R) 4\pi R^2 = 4\pi\varepsilon_o AR^4$$

b) $\overline{E} = Ar^2 \overline{i}_r \rightarrow \nabla \cdot \overline{E} = \frac{1}{r} \frac{\partial}{\partial r} (rE_r) = 3Ar = \frac{\rho}{\varepsilon_o}$

$$Q_T = \int_V \rho \, dV$$

$$= \int_V 3\varepsilon_o A r^2 dr d\phi dz$$

$$= 6\pi\varepsilon_o AL \int_0^a r^2 dr$$

$$= 2\pi\varepsilon_o ALa^3$$

c) $\overline{E} = A(x\overline{i}_x + y\overline{i}_y) \rightarrow \nabla\cdot\overline{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 2A = \frac{\rho}{\varepsilon_o}$

$$Q_T = \rho\, a^3 = 2\varepsilon_o Aa^3$$

23. a) $\varepsilon_o \frac{dE_x}{dx} = \rho(x) = \begin{cases} \rho_o e^{-x/a} & x > 0 \\ \rho_o e^{+x/a} & x < 0 \end{cases}$

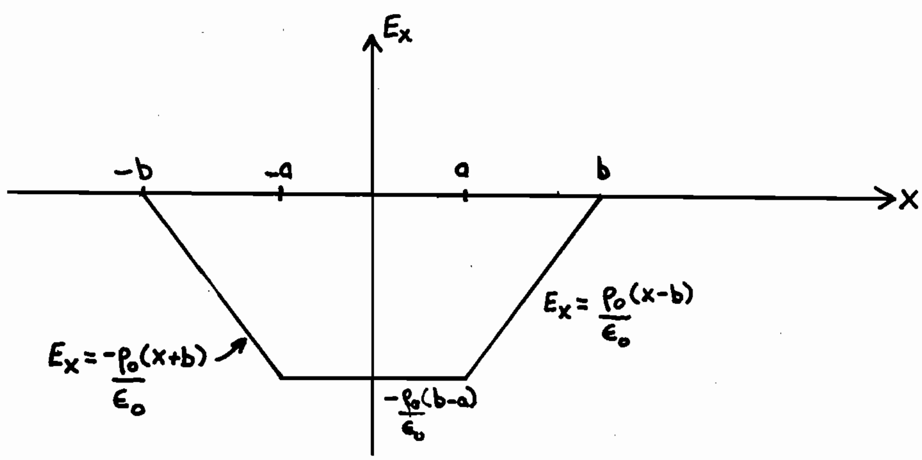
Integrating

$$E_x = \begin{cases} \frac{-\rho_o a}{\varepsilon_o} e^{-x/a} + C_1 & x > 0 \\ \frac{\rho_o a}{\varepsilon_o} e^{+x/a} + C_2 & x < 0 \end{cases}$$

By symmetry, $E_x(x=0) = 0 \rightarrow C_1 = \frac{\rho_o a}{\varepsilon_o}$, $C_2 = \frac{-\rho_o a}{\varepsilon_o}$

$$E_x = \begin{cases} \frac{-\rho_o a}{\varepsilon_o} (e^{-x/a} - 1) & x > 0 \\ \frac{\rho_o a}{\varepsilon_o} (e^{x/a} - 1) & x < 0 \end{cases}$$

b)



c) $\varepsilon_o \dfrac{dE_x}{dx} = \rho(x) = \dfrac{\rho_o x}{d} \rightarrow E_x = \begin{cases} \dfrac{\rho_o}{2\varepsilon_o d}(x^2 - d^2) & |x| < d \\ 0 & |x| > d \end{cases}$

d) $\rho(x) = \varepsilon_o \dfrac{dE_x}{dx} = \begin{cases} \rho_o(1 + \frac{x}{d}) & -d < x < 0 \\ \rho_o(1 - \frac{x}{d}) & 0 < x < d \\ 0 & |x| > d \end{cases}$

Integrating

$$E_x = \begin{cases} \dfrac{\rho_o x}{\varepsilon_o}\left(1 + \dfrac{x}{2d}\right) & -d < x < 0 \\ \dfrac{\rho_o x}{\varepsilon_o}\left(1 - \dfrac{x}{2d}\right) & 0 < x < d \\ \pm \dfrac{\rho_o d}{2\varepsilon_o} & \begin{matrix} x > d \\ x < -d \end{matrix} \end{cases}$$

24. a) $\dfrac{\varepsilon_o}{r^2} \dfrac{\partial}{\partial r}(r^2 E_r) = \rho(r) = \rho_o e^{-r/a}$

$$\frac{\partial}{\partial r}(r^2 E_r) = \frac{\rho_o r^2}{\varepsilon_o} e^{-r/a} \rightarrow r^2 E_r = \frac{-\rho_o a}{\varepsilon_o}\left[r^2 + 2a^2\left(\frac{r}{a} + 1\right)\right]e^{-r/a} + \text{constant}$$

No point charge at $r = 0 \rightarrow \text{constant} = \dfrac{2a^3 \rho_o}{\varepsilon_o}$

$$E_r = \frac{-\rho_o a}{\varepsilon_o}\left\{\left[1 + \frac{2a}{r} + \frac{2a^2}{r^2}\right] e^{-r/a} - \frac{2a^2}{r^2}\right\}$$

b) Use Gaussian sphere of radius r

$$\oint_S \varepsilon_o \overline{E}\cdot\overline{dS} = 4\pi\varepsilon_o r^2 E_r = Q_{\text{Tenclosed}} = \begin{cases} \frac{4}{3}\pi r^3 \rho_1 & r < R_1 \\ \frac{4}{3}\pi R_1^3 \rho_1 + \frac{4}{3}\pi(r^3 - R_1^3)\rho_2 & R_1 < r < R_2 \\ \frac{4}{3}\pi[\rho_1 R_1^3 + (R_2^3 - R_1^3)\rho_2] & r > R_2 \end{cases}$$

$$E_r = \begin{cases} \dfrac{\rho_1 r}{3\varepsilon_o} & r < R_1 \\ \dfrac{\rho_1 R_1^3 + (r^3 - R_1^3)\rho_2}{3\varepsilon_o r^2} & R_1 < r < R_2 \\ \dfrac{[\rho_1 R_1^3 + (R_2^3 - R_1^3)\rho_2]}{3\varepsilon_o r^2} & r > R_2 \end{cases}$$

c) Use Gaussian sphere of radius r

$$\oint_S \varepsilon_o \overline{E}\cdot\overline{dS} = 4\pi\varepsilon_o r^2 E_r = Q_{Tenclosed} = \begin{cases} \dfrac{4\pi\rho_o}{R}\displaystyle\int_0^r r'^3 dr' = \dfrac{\pi\rho_o r^4}{R} & r < R \\ \dfrac{4\pi\rho_o}{R}\displaystyle\int_0^R r'^3 dr' = \pi\rho_o R^3 & r > R \end{cases}$$

$$E_r = \begin{cases} \dfrac{\rho_o r^2}{4\varepsilon_o R} & r < R \\ \dfrac{\rho_o R^3}{4\varepsilon_o r^2} & r > R \end{cases}$$

25. a) $$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) = \frac{\rho_o}{\varepsilon_o} e^{-r/a}$$

$$rE_r = \frac{\rho_o}{\varepsilon_o}\int re^{-r/a} dr + \text{constant}$$

$$= \frac{\rho_o a^2}{\varepsilon_o} e^{-r/a}(-\frac{r}{a} - 1) + \text{constant}$$

$$E_r = \frac{-\rho_o a^2}{\varepsilon_o}\left[e^{-r/a}(\frac{1}{a} + \frac{1}{r}) - \frac{1}{r}\right] \qquad (E_r(r=0) \text{ finite})$$

b) $$2\pi\varepsilon_o r E_r = \lambda_{Tenclosed} = \begin{cases} \pi r^2 \rho_1 & r < a \\ \pi a^2 \rho_1 + \pi(r^2 - a^2)\rho_2 & a < r < b \\ \pi a^2 \rho_1 + \pi(b^2 - a^2)\rho_2 & r > b \end{cases}$$

$$E_r = \begin{cases} \dfrac{\rho_1 r}{2\varepsilon_o} & r < a \\[2ex] \dfrac{\rho_1 a^2 + (r^2 - a^2)\rho_2}{2\varepsilon_o r} & a < r < b \\[2ex] \dfrac{\rho_1 a^2 + (b^2 - a^2)\rho_2}{2\varepsilon_o r} & r > b \end{cases}$$

c) $$2\pi\varepsilon_o r E_r = \begin{cases} \dfrac{2\pi\rho_o}{a}\displaystyle\int_0^r r'^2 dr' = \dfrac{2\pi\rho_o r^3}{3a} & r < a \\[2ex] \dfrac{2\pi\rho_o}{a}\displaystyle\int_0^a r'^2 dr' = \dfrac{2\pi\rho_o a^2}{3} & r > a \end{cases}$$

$$E_r = \begin{cases} \dfrac{\rho_o r^2}{3\varepsilon_o a} & r < a \\[2ex] \dfrac{\rho_o a^2}{3\varepsilon_o r} & r > a \end{cases}$$

26. If the hole is filled with uniform charge density ρ_o, the electric field within the cylinder is

$$\overline{E} = \frac{\rho_o r \overline{i}_r}{2\varepsilon_o} = \frac{\rho_o}{2\varepsilon_o}(x\overline{i}_x + y\overline{i}_y) \qquad r < R$$

If the hole is filled with uniform charge density $-\rho_o$, the electric field within the hole is

$$\overline{E}' = \frac{-\rho_o r' \overline{i}_{r'}}{2\varepsilon_o} = \frac{-\rho_o}{2\varepsilon_o}[(x - d)\overline{i}_x + y\overline{i}_y] \qquad r' < b$$

where the primed cylindrical coordinate terms are measured with respect to an axis through the center of the hole. The total electric field within the hole is

$$\overline{E}_T = \overline{E} + \overline{E}' = \frac{\rho_o d}{2\varepsilon_o}\overline{i}_x$$

27. $$E_z = -\frac{\partial V}{\partial z} = \frac{\sigma_o}{2\varepsilon_o} \rightarrow V = \frac{-\sigma_o z}{2\varepsilon_o} + \text{constant}$$

Each incremental charge element λdy at coordinate y, when rotated winds up at position $z = (\ell - y)$ which requires incremental work

$$dW = \lambda dy[V(z) - V(z=0)]$$

$$= \frac{-\lambda\sigma_o}{2\varepsilon_o}(\ell - y)dy$$

The total work required is

$$W = \frac{-\lambda\sigma_o}{2\varepsilon_o}\int_0^{\ell}(\ell - y)dy$$

$$= \frac{\lambda\sigma_o}{2\varepsilon_o}\frac{(\ell - y)^2}{2}\Bigg|_0^{\ell} = \frac{-\lambda\sigma_o\ell^2}{4\varepsilon_o}$$

28. $$\frac{1}{2}mv^2 + qV = \frac{1}{2}mv^2 + \frac{qQ}{4\pi\varepsilon_o r} = \text{constant} = \frac{1}{2}mv_o^2$$

a) For $v = 0$ at $r = R$

$$v_o \geq \sqrt{\frac{qQ}{2\pi\varepsilon_o Rm}}$$

b) If $v_o = \frac{1}{2}\sqrt{\frac{qQ}{2\pi\varepsilon_o Rm}}$, then $v = 0$ when

$$\frac{qQ}{4\pi\varepsilon_o r} = \frac{1}{16}\frac{qQ}{\pi\varepsilon_o Rm} \rightarrow r = 4R$$

29. a) $V = Ax^2 \rightarrow \overline{E} = -\nabla V = -\frac{\partial V}{\partial x}\overline{i}_x = -2Ax\overline{i}_x$

$$\rho = \varepsilon_o\nabla\cdot\overline{E} = \varepsilon_o\frac{\partial E_x}{\partial x} = -2A\varepsilon_o$$

b) $V = Axyz \rightarrow \overline{E} = -\nabla V = -\frac{\partial V}{\partial x}\overline{i}_x - \frac{\partial V}{\partial y}\overline{i}_y - \frac{\partial V}{\partial z}\overline{i}_z = -A[yz\overline{i}_x + xz\overline{i}_y + xy\overline{i}_z]$

$$\rho = \varepsilon_o\nabla\cdot\overline{E} = 0$$

c) $V = Ar^2\sin\phi + Brz$

$$\overline{E} = -\nabla V = -\frac{\partial V}{\partial r}\overline{i}_r - \frac{1}{r}\frac{\partial V}{\partial\phi}\overline{i}_\phi - \frac{\partial V}{\partial z}\overline{i}_z$$

$$= -[(2Ar\sin\phi + Bz)\overline{i}_r + Ar\cos\phi\,\overline{i}_\phi + Br\overline{i}_z]$$

$$\rho = \varepsilon_o\nabla\cdot\overline{E} = \varepsilon_o\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right]$$

$$= -\varepsilon_o[3A\sin\phi + \frac{Bz}{r}]$$

d) $V = Ar^2\sin\theta\cos\phi$

$$\bar{E} = -\nabla V = -[\frac{\partial V}{\partial r}\bar{i}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{i}_\phi]$$

$$= -A[2r\sin\theta\cos\phi\bar{i}_r + r\cos\theta\cos\phi\bar{i}_\theta - r\sin\phi\bar{i}_\phi]$$

$$\rho = \varepsilon_o\nabla\cdot\bar{E} = \varepsilon_o[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(E_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial \phi}]$$

$$= -\varepsilon_o A\,[6\sin\theta\cos\phi + \frac{\cos\phi}{\sin\theta}\cos 2\theta - \frac{\cos\phi}{\sin\theta}]$$

30. $\nabla \times \bar{E}$ must be zero.

a) $\bar{E} = ax^2y^2\bar{i}_x$

$$\nabla \times \bar{E} = \det\begin{bmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2y^2 & 0 & 0 \end{bmatrix} = -\bar{i}_z\frac{\partial}{\partial y}(ax^2y^2) = -2ayx^2\bar{i}_z \neq 0$$

Not an electric field!

b) $\bar{E} = a(\bar{i}_r\cos\theta - \bar{i}_\theta\sin\theta) = a\bar{i}_z$ (uniform electric field)

$$\nabla\cdot\bar{E} = \frac{\rho}{\varepsilon_o} = 0$$

c) $\bar{E} = a(y\bar{i}_x - x\bar{i}_y)$

$$\nabla \times \bar{E} = \bar{i}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -2a\bar{i}_z \neq 0 \quad \text{(not an electric field)}$$

d) $\bar{E} = \frac{a}{r^2}[\bar{i}_r(1 + \cos\phi) + \sin\phi\bar{i}_\phi]$

$$\nabla \times \bar{E} = \frac{1}{r}[\frac{\partial}{\partial r}(rE_\phi) - \frac{\partial E_r}{\partial \phi}]\bar{i}_z$$

$$= \frac{1}{r}[\frac{-\sin\phi}{r^2} + \frac{\sin\phi}{r^2}] = 0 \quad \text{(can be an electric field)}$$

$$\nabla\cdot\bar{E} = \frac{\rho}{\varepsilon_o} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi}$$

$$= \frac{-a}{r^3}(1 + \cos\phi) + \frac{a}{r^3}\cos\phi = \frac{-a}{r^3} \rightarrow \rho = -\frac{\varepsilon_o a}{r^3}$$

31. a) $E_x = \dfrac{\sigma_o}{\varepsilon_o}$ (between charged sheets)

$$\overline{E} = -\nabla V \rightarrow V = \frac{-\sigma_o x}{\varepsilon_o} + \text{constant}$$

$$\Delta v = V(x=0) - V(x=a) = \frac{\sigma_o a}{\varepsilon_o}$$

b) $$E_r = -\frac{\partial V}{\partial r} = \frac{\lambda_o}{2\pi\varepsilon_o r} \rightarrow V = \frac{-\lambda_o}{2\pi\varepsilon_o}\ell n r + \text{constant}$$

$$\Delta v = V(r = a) - V(r = b) = \frac{\lambda_o}{2\pi\varepsilon_o}\ell n\frac{b}{a}$$

c) $$E_r = \frac{q_o}{4\pi\varepsilon_o r^2} = -\frac{\partial V}{\partial r} \rightarrow V = \frac{q_o}{4\pi\varepsilon_o r} + \text{constant}$$

$$\Delta v = V(r=R_1) - V(r=R_2) = \frac{q_o}{4\pi\varepsilon_o}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

32. a) $dq = \dfrac{Q}{2\pi R^2} 2\pi R\sin\theta R d\theta = Q\sin\theta d\theta$

$$z' = R\cos\theta;\ dz' = -R\sin\theta d\theta \rightarrow dq = -\frac{Q}{R}dz'$$

$$\text{radius of hoop } r = \sqrt{R^2 - z'^2}$$

b) $$dV = \frac{dq}{4\pi\varepsilon_o\sqrt{(z - z')^2 + R^2\sin^2\theta}} = \frac{-Q\,dz'}{4\pi\varepsilon_o R\sqrt{(z - z')^2 + R^2 - z'^2}}$$

$$= \frac{-Q\,dz'}{4\pi\varepsilon_o R[z^2 - 2zz' + R^2]^{1/2}}$$

c) $$V = \frac{-Q}{4\pi\varepsilon_o R}\int_{z'=R}^{0}\frac{dz'}{[z^2 - 2zz' + R^2]^{1/2}}$$

$$= \frac{-Q}{4\pi\varepsilon_o R}\left.\frac{2\sqrt{z^2 - 2zz' + R^2}}{(-2z)}\right|_{z'=R}^{0}$$

$$= \frac{-Q}{4\pi\varepsilon_o Rz}\left[|z - R| - \sqrt{z^2 + R^2}\right]$$

$$V = \begin{cases} \dfrac{-Q}{4\pi\varepsilon_o Rz}\left[z - R - \sqrt{z^2 + R^2}\right] & z > R \\ \dfrac{-Q}{4\pi\varepsilon_o Rz}\left[R - z - \sqrt{z^2 + R^2}\right] & z < R \end{cases}$$

Check $z \to + \infty$

$$V \approx \frac{-Q}{4\pi\varepsilon_o Rz}\left[z - R - z(1 + \frac{1}{2}\frac{R^2}{z^2})\right]$$

$$\approx \frac{Q}{4\pi\varepsilon_o z}$$

$z \to - \infty$

$$V \approx \frac{-Q}{4\pi\varepsilon_o Rz}\left[R - z + z(1 + \frac{1}{2}\frac{R^2}{z^2})\right]$$

$$\approx \frac{-Q}{4\pi\varepsilon_o Rz}$$

d) $z > R$

$$E_z = -\frac{\partial V}{\partial z} = \frac{Q}{4\pi\varepsilon_o R}\frac{\partial}{\partial z}\left\{1 - \frac{R}{z} - \frac{\sqrt{z^2 + R^2}}{z}\right\}$$

$$= \frac{Q}{4\pi\varepsilon_o R}\left\{\frac{R}{z^2} + \frac{\sqrt{z^2 + R^2}}{z^2} - \frac{1}{\sqrt{z^2 + R^2}}\right\}$$

$$= \frac{Q}{4\pi\varepsilon_o R}\left\{\frac{R}{z^2} + \frac{R^2}{z^2\sqrt{z^2 + R^2}}\right\}$$

$$= \frac{Q}{4\pi\varepsilon_o z^2}\left\{1 + \frac{R}{\sqrt{z^2 + R^2}}\right\}$$

$z < R$

$$E_z = -\frac{\partial V}{\partial z} = \frac{Q}{4\pi\varepsilon_o R}\frac{\partial}{\partial z}\left[\frac{R}{z} - 1 - \frac{\sqrt{z^2 + R^2}}{z}\right]$$

$$= \frac{Q}{4\pi\varepsilon_o R}\left[\frac{-R}{z^2} + \frac{\sqrt{z^2 + R^2}}{z^2} - \frac{1}{\sqrt{z^2 + R^2}}\right]$$

$$= \frac{Q}{4\pi\varepsilon_o R}\left[\frac{-R}{z^2} + \frac{R^2}{z^2\sqrt{z^2 + R^2}}\right]$$

$$= \frac{Q}{4\pi\varepsilon_o z^2}\left[-1 + \frac{R}{\sqrt{z^2 + R^2}}\right]$$

Check:

$$\sigma_f = \varepsilon_o[E_z(z=R_+) - E_z(z=R_-)] = \frac{Q}{2\pi R^2}$$

e) $\sigma_o = \frac{Q}{2\pi R^2} \rightarrow \rho_o dr$

$$dV = \begin{cases} -\frac{\rho_o r dr}{2\varepsilon_o z}[z - r - \sqrt{z^2 + r^2}] & z > r \\ -\frac{\rho_o r dr}{2\varepsilon_o z}[r - z - \sqrt{z^2 + r^2}] & z < r \end{cases}$$

$z \geq R$

$$V = \frac{-\rho_o}{2\varepsilon_o z}\int_{r=0}^{R}[rz - r^2 - r\sqrt{z^2 + r^2}]dr$$

$$= \frac{-\rho_o}{2\varepsilon_o z}\left[\frac{R^2 z}{2} - \frac{R^3}{3} - \frac{1}{3}[R^2 + z^2]^{3/2} + \frac{z^3}{3}\right]$$

$z \leq 0$

$$V = \frac{-\rho_o}{2\varepsilon_o z}\int_{r=0}^{R}[r^2 - zr - r\sqrt{z^2 + r^2}]dr$$

$$= \frac{-\rho_o}{2\varepsilon_o z}\left[\frac{R^3}{3} - \frac{zR^2}{2} - \frac{1}{3}[R^2 + z^2]^{3/2} + \frac{z^3}{3}\right]$$

$0 \leq z \leq R$

$$V = \frac{-\rho_o}{2\varepsilon_o z}\left[\int_0^z [rz - r^2 - r\sqrt{z^2 + r^2}]dr + \int_z^R [r^2 - zr - r\sqrt{z^2 + r^2}]dr\right]$$

$$= \frac{-\rho_o}{2\varepsilon_o z}\left[\frac{z^3}{2} - \frac{z^3}{3} - \frac{1}{3}[2z^2]^{3/2} + \frac{z^3}{3}\right.$$

$$\left. + \frac{R^3 - z^3}{3} - \frac{z}{2}(R^2 - z^2) - \frac{1}{3}(R^2 + z^2)^{3/2} + \frac{1}{3}(2z^2)^{3/2}\right]$$

$$= \frac{-\rho_o}{2\varepsilon_o z}\left[\left[\frac{R^3}{3} - \frac{R^2 z}{2} - \frac{1}{3}(R^2 + z^2)^{3/2} + \frac{2}{3}z^3\right]\right]$$

$$E_z = -\frac{\partial V}{\partial z} = \begin{cases} \frac{\rho_o}{2\varepsilon_o}\left[\frac{R^3}{3z^2} + \frac{2}{3}z - \frac{1}{3}[R^2 + z^2]^{1/2}[2 - \frac{R^2}{z^2}]\right] & z \geq R \\ \frac{\rho_o}{2\varepsilon_o}\left[-\frac{R^3}{3z^2} + \frac{2z}{3} - \frac{1}{3}[R^2 + z^2]^{1/2}[2 - \frac{R^2}{z^2}]\right] & z \leq 0 \\ \frac{\rho_o}{2\varepsilon_o}\left[-\frac{R^3}{3z^2} + \frac{4z}{3} - \frac{1}{3}[R^2 + z^2]^{1/2}[2 - \frac{R^2}{z^2}]\right] & 0 \leq z \leq R \end{cases}$$

33. $V = \frac{1}{4\pi\varepsilon_o}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2}\right]$

$$r_1 = [r^2 + (\tfrac{a}{2})^2 - ar\cos\theta]^{1/2}; \quad r_2 = [r^2 + (\tfrac{a}{2})^2 + ar\cos\theta]^{1/2}$$

a) $$V = \frac{1}{4\pi\varepsilon_o}\left\{\frac{q_1}{[r^2 + (\frac{a}{2})^2 - ar\cos\theta]^{1/2}} + \frac{q_2}{[r^2 + (\frac{a}{2})^2 + ar\cos\theta]^{1/2}}\right\}$$

b) $$\overline{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\overline{i}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\overline{i}_\theta\right]$$

$$= -\frac{1}{4\pi\varepsilon_o}\left\{\frac{q_1\left[-\frac{1}{2}[(2r - a\cos\theta)\overline{i}_r + a\sin\theta\,\overline{i}_\theta]\right]}{[r^2 + (\frac{a}{2})^2 - ar\cos\theta]^{3/2}} + \frac{q_2\left[-\frac{1}{2}[(2r + a\cos\theta)\overline{i}_r - a\sin\theta\,\overline{i}_\theta]\right]}{[r^2 + (\frac{a}{2})^2 + ar\cos\theta]^{3/2}}\right\}$$

$$= \frac{1}{8\pi\varepsilon_o}\left\{\frac{q_1[(2r - a\cos\theta)\overline{i}_r + a\sin\theta\,\overline{i}_\theta]}{[r^2 + (\frac{a}{2})^2 - ar\cos\theta]^{3/2}} + \frac{q_2[(2r + a\cos\theta)\overline{i}_r - a\sin\theta\,\overline{i}_\theta]}{[r^2 + (\frac{a}{2})^2 + ar\cos\theta]^{3/2}}\right\}$$

c) $\lim r \gg a, \; q_1 = -q_2 \equiv q_o$

$$[r^2 + (\tfrac{a}{2})^2 - ar\cos\theta] \approx r^2[1 + \frac{a^2}{4r^2} - \frac{a}{r}\cos\theta] \approx r^2(1 - \frac{a}{r}\cos\theta)$$

$$r_1 \approx r(1 - \frac{1}{2}\frac{a}{r}\cos\theta); \quad r_2 \approx r(1 + \frac{1}{2}\frac{a}{r}\cos\theta)$$

$$V \approx \frac{q_o}{4\pi\varepsilon_o r}\left\{(1 + \frac{1}{2}\frac{a}{r}\cos\theta) - (1 - \frac{1}{2}\frac{a}{r}\cos\theta)\right\}$$

$$= \frac{q_o a}{4\pi\varepsilon_o r^2}\cos\theta$$

$$\overline{E} \approx \frac{q_o}{8\pi\varepsilon_o}\left\{\frac{(1 + \frac{3a}{2r}\cos\theta)}{r^3}[(2r - a\cos\theta)\overline{i}_r + a\sin\theta\overline{i}_\theta] - \frac{(1 - \frac{3a}{2r}\cos\theta)}{r^3}[(2r + a\cos\theta)\overline{i}_r - a\sin\theta\overline{i}_\theta]\right\}$$

$$\approx \frac{q_o}{8\pi\varepsilon_o r^3}\{4a\cos\theta\overline{i}_r + 2a\sin\theta\overline{i}_\theta\}$$

$$\approx \frac{q_o a}{4\pi\varepsilon_o r^3}[2\cos\theta\overline{i}_r + \sin\theta\overline{i}_\theta]$$

d) $\frac{dr}{rd\theta} = \frac{E_r}{E_\theta} = 2\cot\theta \rightarrow \int \frac{dr}{r} = \int 2\cot\theta d\theta \rightarrow \ell nr = 2\ell n\sin\theta + \text{constant} \rightarrow r = r_o\sin^2\theta$

34. a) $V_1 = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_2}{r_{12}} + \frac{q_3}{r_{13}}\right)$

$V_2 = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1}{r_{12}} + \frac{q_3}{r_{23}}\right)$

$V_3 = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$

b) $V_1' = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_2'}{r_{12}} + \frac{q_3'}{r_{13}}\right)$

$V_2' = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1'}{r_{12}} + \frac{q_3'}{r_{23}}\right)$

$V_3' = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1'}{r_{13}} + \frac{q_2'}{r_{23}}\right)$

$$q_1' V_1 + q_2' V_2 + q_3' V_3 = q_1 V_1' + q_2 V_2' + q_3 V_3'$$

c) By breaking each conductor into many incremental point charges, because each are at the same potential (the conductor is an equipotential surface) the potential may be taken outside the summation

$$\underset{\text{on conductor}}{\Sigma}(q_i V_i' - q_i' V_i) = V_i' \underbrace{\Sigma q_i} - V_i \underbrace{\Sigma q_i'}$$

Total charge on conductor

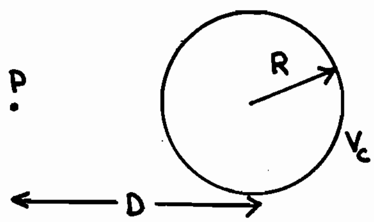
d) Let $q_1 = q$, $q_2 = q_c$, $V_2 = 0$

$q_1' = 0$, $V_1' = V_p$, $V_2' = V_c$

$$\cancel{q_1'}\,\cancelto{0}{V_1} + \cancel{q_2'}\,\cancelto{0}{V_2} = \underbrace{q_1\,V_1'}_{qV_p} + \underbrace{q_2\,V_2'}_{q_cV_c}$$

$$qV_p + q_cV_c = 0 \to q_c = -\frac{qV_p}{V_c}$$

e)



$$V_p = \frac{V_cR}{D}, \quad q_c = -\frac{qR}{D}$$

f) $$V = C_1 \ln r/r_o \to V_c = C_1 \ln \frac{a}{r_o} \to C_1 = \frac{V_c}{\ln \frac{a}{r_o}} \quad (r_o \to \infty)$$

$$V_p = \frac{V_c}{\ln \frac{a}{r_o}} \ln \frac{D}{r_o} \to \lambda_c = -\frac{\lambda V_p}{V_c} = -\lambda \frac{\ln D/r_o}{\ln a/r_o}$$

$$\lim_{r_o \to \infty} \lambda_c = -\lambda$$

g) $q_1 = q$, $q_2 = q(y{=}0)$, $q_3 = q(y{=}d)$, $V_2 = 0$, $V_3 = 0$

$q_1' = 0$, $V_2' = V_o$, $V_3' = 0$

$$\underbrace{q_1\,V_1'}_{qV_1'} + \underbrace{q_2V_2'}_{q(y=0)V_o} + \cancel{q_3}\cancelto{0}{V_3'} = \cancel{q_1'}\cancelto{0}{V_1} + \cancel{q_2'}\cancelto{0}{V_2} + \cancel{q_3'}\cancelto{0}{V_3}$$

$$qV_1' + q(y{=}0)V_o = 0$$

$$V_1' = V_o(1 - \tfrac{a}{d})$$

$$q(y{=}0) = -\frac{qV_1'}{V_o} = -q(1 - \tfrac{a}{d}) = -\frac{qb}{d}$$

$$q(y{=}d) = -q - q(y{=}0) = -q[1 - \tfrac{b}{d}] = -\frac{qa}{d}$$

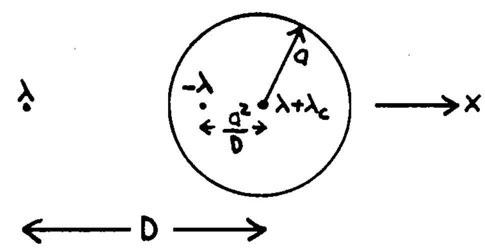
h) $q(y=0) = -\frac{q}{d}(d - x)$

$q(y=d) = -\frac{qx}{d}$

$$i = \frac{dq(y=0)}{dt} = -\frac{dq(y=d)}{dt} = \frac{q}{d}\frac{dx}{dt} = \frac{qv_o}{d}$$

Section 2.6

35. $$f_x = \frac{\lambda}{2\pi\varepsilon_o}\left[\frac{-\lambda}{D - \frac{a^2}{D}} + \frac{(\lambda + \lambda_c)}{D}\right]$$



36. a) $$dE_y = \frac{\sigma_o dy'}{2\pi\varepsilon_o(y - y')} \qquad y > y'$$

$y > d$

$$E_y = \frac{\sigma_o}{2\pi\varepsilon_o}\int_{y'=0}^{d}\frac{dy'}{(y - y')}$$

$$= \frac{-\sigma_o}{2\pi\varepsilon_o}\ln(y - y')\Big|_{y'=0}^{d} = \frac{-\sigma_o}{2\pi\varepsilon_o}\ln\left(1 - \frac{d}{y}\right)$$

$0 < y < d$

$$E_y = \frac{\sigma_0}{2\pi\varepsilon_0}\left[\int_0^y \frac{dy'}{(y-y')} - \int_y^d \frac{dy'}{(y'-y)}\right]$$

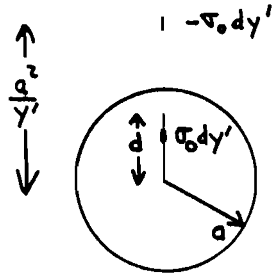
$$= \frac{\sigma_0}{2\pi\varepsilon_0}\ln\frac{y}{d-y}$$

$y < 0$

$$E_y = -\frac{\sigma_0}{2\pi\varepsilon_0}\int_{y'=0}^{d}\frac{dy'}{(y'-y)}$$

$$= -\frac{\sigma_0}{2\pi\varepsilon_0}\ln\left(1 - \frac{d}{y}\right)$$

b)



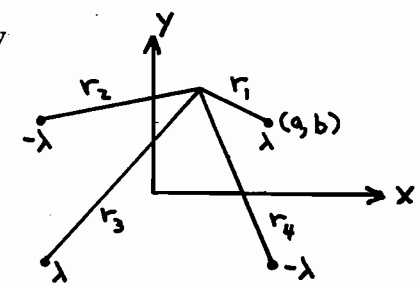
c) $$d\overline{F} = \overline{i}_y\,\sigma_o dy'\,E_y\left(y = \frac{a^2}{y'}\right)$$

$$= \overline{i}_y\,\sigma_o dy'\left(\frac{-\sigma_o}{2\pi\varepsilon_o}\right)\ln\left(1 - \frac{dy'}{a^2}\right)$$

$$\overline{F} = \frac{-\sigma_o^2}{2\pi\varepsilon_o}\,\overline{i}_y\int_{y'=0}^{d}\ln\left(1 - \frac{dy'}{a^2}\right)dy'$$

$$= \frac{-\sigma_o^2 a^2}{2\pi\varepsilon_o d}\left(1 - \frac{dy'}{a^2}\right)\left[\ln\left(1 - \frac{dy'}{a^2}\right) - 1\right]\Big|_{y'=0}^{d}\,\overline{i}_y$$

$$= \frac{\sigma_o^2 a^2}{2\pi\varepsilon_o d}\left\{(1 - \frac{d^2}{a^2})[\ln(1 - \frac{d^2}{a^2}) - 1] + 1\right\} \bar{i}_y$$



37. a) $V = \frac{-\lambda}{2\pi\varepsilon_o}[\ln r_1 - \ln r_2 + \ln r_3 - \ln r_4]$

$$r_1 = [(y - b)^2 + (x - a)^2]^{1/2};\ r_2 = [(y - b)^2 + (x + a)^2]^{1/2}$$

$$r_3 = [(y + b)^2 + (x + a)^2]^{1/2};\ r_4 = [(y + b)^2 + (x - a)^2]^{1/2}$$

$$\bar{E} = -\nabla V = \frac{\lambda}{2\pi\varepsilon_o}\left\{\frac{[(y - b)\bar{i}_y + (x - a)\bar{i}_x]}{r_1^2} - \frac{[(y - b)\bar{i}_y + (x + a)\bar{i}_x]}{r_2^2}\right.$$

$$\left. + \frac{[(y + b)\bar{i}_y + (x + a)\bar{i}_x]}{r_3^2} - \frac{[(y + b)\bar{i}_y + (x - a)\bar{i}_x]}{r_4^2}\right\}$$

$E_x(y=0) = 0,\ E_y(x=0) = 0$

b) At $x = a,\ y = b$

$$r_1 = 0,\ r_2 = 2a,\ r_3 = \sqrt{(2a)^2 + (2b)^2},\ r_4 = 2b$$

$$\bar{F} = \frac{\lambda^2}{2\pi\varepsilon_o}\left[\frac{-\bar{i}_x}{2a} - \frac{\bar{i}_y}{2b} + \frac{(a\bar{i}_x + b\bar{i}_y)}{2(a^2 + b^2)}\right]$$

$$= \frac{\lambda^2}{4\pi\varepsilon_o}\left[-\bar{i}_x(\frac{1}{a} - \frac{a}{a^2 + b^2}) - \bar{i}_y(\frac{1}{b} - \frac{b}{a^2 + b^2})\right]$$

$$= \frac{-\lambda^2}{4\pi\varepsilon_o(a^2 + b^2)ab}\{b^3\bar{i}_x + a^3\bar{i}_y\}$$

c) $\sigma_f(x=0) = \varepsilon_o E_x(x=0) = \frac{\lambda}{2\pi}\left(\frac{2a}{r_3^2} - \frac{2a}{r_1^2}\right) = \frac{\lambda a}{\pi}\left(\frac{1}{[(y+b)^2 + a^2]} - \frac{1}{[(y-b)^2 + a^2]}\right)$

$\sigma_f(y=0) = \varepsilon_o E_y(y=0) = \frac{\lambda}{2\pi}\left(\frac{-2b}{r_1^2} + \frac{2b}{r_2^2}\right) = \frac{\lambda b}{\pi}\left(\frac{1}{[(x-a)^2 + b^2]} - \frac{1}{[(x+a)^2 + b^2]}\right)$

$$\lambda(x=0) = \int_0^\infty \sigma_f(x=0)\,dy$$

$$= \frac{\lambda a}{\pi}\int_0^\infty \left[\frac{1}{[(y+b)^2+a^2]} - \frac{1}{[(y-b)^2+a^2]}\right] dy$$

$$= \frac{\lambda a}{\pi}\left\{\frac{1}{a}\tan^{-1}\frac{y+b}{a} - \frac{1}{a}\tan^{-1}\frac{y-b}{a}\right\}\Big|_0^\infty$$

$$= \frac{\lambda}{\pi}\left\{\frac{\pi}{2} - \tan^{-1}\frac{b}{a} - \frac{\pi}{2} - \tan^{-1}\frac{b}{a}\right\}$$

$$= -\frac{2\lambda}{\pi}\tan^{-1}\frac{b}{a}$$

$$\lambda(y=0) = -\frac{2\lambda}{\pi}\tan^{-1}\frac{a}{b}$$

$$\lambda(x=0) + \lambda(y=0) = -\frac{2\lambda}{\pi}\underbrace{\left[\tan^{-1}\frac{a}{b} + \tan^{-1}\frac{b}{a}\right]}_{\pi/2}$$

$$= -\lambda$$

d) Use single image charge at (a,-b)

$$\overline{F} = \frac{\lambda^2}{2\pi\varepsilon_o}\left[+\frac{\overline{i}_y}{2b} - \frac{\overline{i}_x}{2a} + \frac{(a\overline{i}_x - b\overline{i}_y)}{2(a^2+b^2)}\right]$$

$$= \frac{\lambda^2}{4\pi\varepsilon_o}\left[-\overline{i}_x\left(\frac{1}{a} - \frac{a}{(a^2+b^2)}\right) + \overline{i}_y\left(\frac{1}{b} - \frac{b}{a^2+b^2}\right)\right]$$

$$= \frac{\lambda^2}{4\pi\varepsilon_o ab(a^2+b^2)}\left[-b^3\overline{i}_x + a^3\overline{i}_y\right]$$

e) $$V = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right]$$

$$r_1 = [(x-a)^2 + (y-b)^2 + z^2]^{1/2};\quad r_2 = [(x+a)^2 + (y-b)^2 + z^2]^{1/2}$$

$$r_3 = [(x+a)^2 + (y+b)^2 + z^2]^{1/2};\quad r_4 = [(x-a)^2 + (y+b)^2 + z^2]^{1/2}$$

$$\overline{E} = -\nabla V = \frac{q}{4\pi\varepsilon_o}\left[\frac{[(x-a)\overline{i}_x + (y-b)\overline{i}_y + z\overline{i}_z]}{r_1^3} - \frac{[(x+a)\overline{i}_x + (y-b)\overline{i}_y + z\overline{i}_z]}{r_2^3}\right.$$

$$\left. + \frac{[(x+a)\overline{i}_x + (y+b)\overline{i}_y + z\overline{i}_z]}{r_3^3} - \frac{[(x-a)\overline{i}_x + (y+b)\overline{i}_y + z\overline{i}_z]}{r_4^3}\right]$$

$$\overline{F} = \frac{q^2}{16\pi\varepsilon_o}\left\{a\left[\frac{1}{(a^2+b^2)^{3/2}} - \frac{1}{a^3}\right]\overline{i}_x + b\left[\frac{1}{(a^2+b^2)^{3/2}} - \frac{1}{b^3}\right]\overline{i}_y\right\}$$

$$\sigma_f(x=0) = \varepsilon_o E_x(x=0), \quad \sigma_f(y=0) = \varepsilon_o E_y(y=0)$$

$$q(x=0) = \int_{y=0}^{\infty}\int_{z=-\infty}^{+\infty} \sigma_f(x=0)\,dydz = \frac{-2q}{\pi}\tan^{-1}\frac{b}{a}$$

$$q(y=0) = \int_{x=0}^{\infty}\int_{z=-\infty}^{+\infty} \sigma_f(y=0)\,dxdz = \frac{-2q}{\pi}\tan^{-1}\frac{a}{b}$$

If charges are outside conducting corner

$$\overline{F} = \frac{q^2}{16\pi\varepsilon_o}\left\{-a\left[\frac{1}{a^3} - \frac{1}{(a^2+b^2)^{3/2}}\right]\overline{i}_x + b\left[\frac{1}{b^3} - \frac{1}{(a^2+b^2)^{3/2}}\right]\overline{i}_y\right\}$$

Section 2.7

38. $F_x = qE_o - \dfrac{q^2}{4\pi\varepsilon_o(2x)^2}$

$$= qE_o - \frac{q^2}{16\pi\varepsilon_o x^2}$$

a) $F_x = 0 \rightarrow x_o^2 = \dfrac{q}{16\pi\varepsilon_o E_o} \rightarrow x_o = \sqrt{\dfrac{q}{16\pi\varepsilon_o E_o}}$

b) $V(x) = -E_o x - \dfrac{q}{16\pi\varepsilon_o x}$

$$V(x_o) = -E_o x_o - \frac{q}{16\pi\varepsilon_o x_o} = -\frac{1}{2}\sqrt{\frac{qE_o}{\pi\varepsilon_o}}$$

$$V\left(\frac{x_o}{2}\right) = -\frac{5}{8}\sqrt{\frac{qE_o}{\pi\varepsilon_o}}$$

$$\frac{1}{2} mv^2 + qV(x) = \text{constant}$$

For $v = 0$ at $x = x_o$

$$\frac{1}{2} mv_o^2 + qV(\frac{x_o}{2}) = 0 + qV(x_o)$$

$$\frac{1}{2} mv_o^2 > q[V(x_o) - V(\frac{x_o}{2})]$$

$$> q \sqrt{\frac{qE_o}{\pi\varepsilon_o}} [-\frac{1}{2} + \frac{5}{8}]$$

$$> \frac{q}{8} \sqrt{\frac{qE_o}{\pi\varepsilon_o}}$$

$$v_o > \frac{1}{2} \sqrt{\frac{q}{m}} [\frac{qE_o}{\pi\varepsilon_o}]^{1/4}$$

c) If $E_o = 0$, $V = \frac{-q}{16\pi\varepsilon_o x}$

$$W = q[V(\infty) - V(d)] = \frac{q^2}{16\pi\varepsilon_o d}$$

39. a) $Q_{ind} = -\frac{R_1}{R_2} Q$

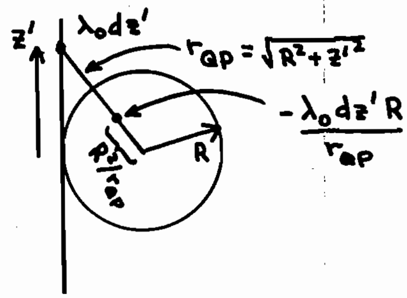
b) $$E_r = \begin{cases} \dfrac{Q_{ind}}{4\pi\varepsilon_o r^2} = -\dfrac{R_1}{R_2} \dfrac{Q}{4\pi\varepsilon_o r^2} & R_1 < r < R_2 \\[2ex] \dfrac{Q + Q_{ind}}{4\pi\varepsilon_o r^2} = \dfrac{(1 - \frac{R_1}{R_2})Q}{4\pi\varepsilon_o r^2} & r > R_2 \end{cases}$$

$$E_r = -\frac{\partial V}{\partial r} \rightarrow V = \begin{cases} \dfrac{-R_1}{R_2} \dfrac{Q}{4\pi\varepsilon_o} (\dfrac{1}{r} - \dfrac{1}{R_1}) & R_1 < r < R_2 \\[2ex] \dfrac{(1 - \frac{R_1}{R_2})Q}{4\pi\varepsilon_o r} & r > R_2 \end{cases}$$

40. Field inside the sphere due to V_o is zero. Only field outside the sphere affected by raising the potential.

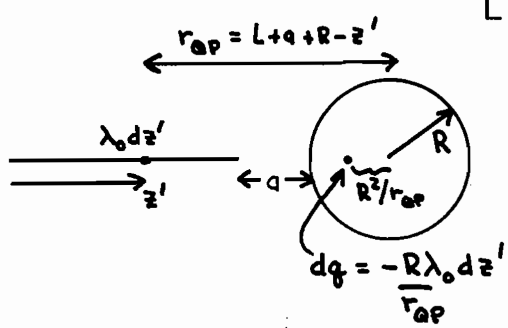
$$F_x = \frac{q\left(-q\,\frac{R}{D}\right)}{4\pi\varepsilon_o\left[\frac{R^2}{D} - D\right]^2} = \frac{-q^2 RD}{4\pi\varepsilon_o\left[R^2 - D^2\right]^2}$$

41. a)



b) $$q_T = \int_{-L/2}^{+L/2} \frac{-\lambda_o R}{\sqrt{R^2 + z'^2}}\,dz' = -\lambda_o R \ln\left[z' + \sqrt{R^2 + z'^2}\right]\Bigg|_{-L/2}^{+L/2}$$

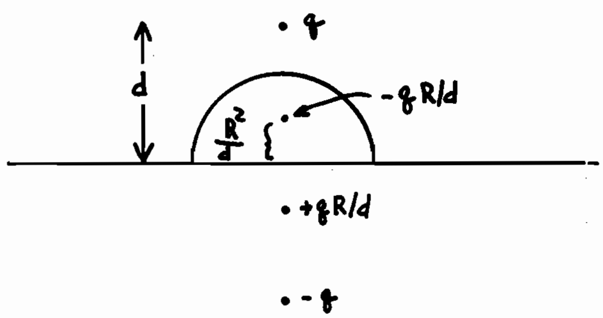
$$= -\lambda_o R \ln\left[\frac{\frac{L}{2} + \sqrt{R^2 + \frac{L^2}{4}}}{-\frac{L}{2} + \sqrt{R^2 + \frac{L^2}{4}}}\right]$$



$$q_T = \int_0^L \frac{-R\lambda_o\,dz'}{L + a + R - z'} = R\lambda_o \ln(L + a + R - z')\Bigg|_0^L$$

$$= R\lambda_o \ln\frac{R + a}{R + a + L}$$

42.



a) $$F_z = q\left[\frac{-qR/d}{4\pi\varepsilon_o(d - \frac{R^2}{d})^2} + \frac{qR/d}{4\pi\varepsilon_o(d + \frac{R^2}{d})^2} - \frac{q}{4\pi\varepsilon_o(2d)^2}\right]$$

$$= \frac{-q^2R}{4\pi\varepsilon_o}\left[\frac{d}{(d^2 - R^2)^2} - \frac{d}{(d^2 + R^2)^2} + \frac{1}{4Rd^2}\right]$$

$$= \frac{-q^2}{4\pi\varepsilon_o}\left[\frac{4R^3d^3}{(d^4 - R^4)^2} + \frac{1}{4d^2}\right]$$

b) $$V = \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{[x^2 + y^2 + (z - d)^2]^{1/2}} - \frac{R/d}{[x^2 + y^2 + (z - \frac{R^2}{d})^2]^{1/2}} + \frac{R/d}{[x^2 + y^2 + (z + \frac{R^2}{d})^2]^{1/2}} - \frac{1}{[x^2 + y^2 + (z + d)^2]^{1/2}}\right\}$$

$$\overline{E} = -\nabla V = \frac{q}{4\pi\varepsilon_o}\left\{\frac{(x\overline{i}_x + y\overline{i}_y + (z-d)\overline{i}_z)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(R/d)(x\overline{i}_x + y\overline{i}_y + (z - \frac{R^2}{d})\overline{i}_z)}{[x^2+y^2+(z - \frac{R^2}{d})^2]^{3/2}} + \frac{(R/d)(x\overline{i}_x + y\overline{i}_y + (z + \frac{R^2}{d})\overline{i}_z)}{[x^2+y^2+(z + \frac{R^2}{d})^2]^{3/2}} - \frac{(x\overline{i}_x + y\overline{i}_y + (z+d)\overline{i}_z)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}$$

At $z = 0$

$$\overline{E} = \overline{i}_z \frac{q}{4\pi\varepsilon_o}\left\{\frac{-2d}{[x^2 + y^2 + d^2]^{3/2}} + \frac{2R^3/d^2}{[x^2 + y^2 + (\frac{R^2}{d})^2]^{3/2}}\right\}$$

$$= \overline{i}_z \frac{qd}{2\pi\varepsilon_o}\left\{\frac{-1}{[r^2 + d^2]^{3/2}} + \frac{R^3/d^3}{[r^2 + (\frac{R^2}{d})^2]^{3/2}}\right\}$$

$$\sigma_f(z=0) = \varepsilon_o E_z(z=0)$$

$$q_T(z=0) = \int_{r=R}^{\infty} 2\pi\sigma_f(z=0)\,r\,dr$$

$$= qd \int_{r=R}^{\infty} \left[\frac{-r}{[r^2 + d^2]^{3/2}} + \frac{r(\frac{R}{d})^3}{[r^2 + (\frac{R^2}{d})^2]^{3/2}} \right] dr$$

$$= qd \left[\frac{1}{\sqrt{r^2 + d^2}} - \frac{(R/d)^3}{\sqrt{r^2 + (\frac{R^2}{d})^2}} \right] \Bigg|_{r=R}^{\infty}$$

$$= qd \left[\frac{(R/d)^3}{\sqrt{R^2 + (R^2/d)^2}} - \frac{1}{\sqrt{R^2 + d^2}} \right]$$

$$= q \frac{(\frac{R^2}{d^2} - 1)}{\sqrt{1 + \frac{R^2}{d^2}}}$$

The easiest way to find the total charge induced on the hemisphere is to realize that the total induced charge is $-q$

$$q_T(r=R) + q_T(z=0) = -q \rightarrow q_T(r=R) = -q \left[1 + \frac{\frac{R^2}{d^2} - 1}{\sqrt{1 + \frac{R^2}{d^2}}} \right]$$

To verify this we convert the Cartesian coordinates to spherical coordinates

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

$$\bar{i}_x = \sin\theta\cos\phi\,\bar{i}_r + \cos\theta\cos\phi\,\bar{i}_\theta - \sin\phi\,\bar{i}_\phi$$

$$\bar{i}_y = \sin\theta\sin\phi\,\bar{i}_r + \cos\theta\sin\phi\,\bar{i}_\theta + \cos\phi\,\bar{i}_\phi$$

$$\bar{i}_z = \cos\theta\,\bar{i}_r - \sin\theta\,\bar{i}_\theta$$

At $r = R$ we have

$$x^2 + y^2 + z^2 = R^2$$

$$x^2 + y^2 + (z \pm d)^2 = R^2 + d^2 \pm 2Rd\cos\theta$$

$$x^2 + y^2 + (z \pm \frac{R^2}{d})^2 = R^2[1 + \frac{R^2}{d^2} \pm \frac{2R}{d}\cos\theta]$$

The radial component of electric field at $r = R$ is then

$$E_r(r{=}R) = \frac{q}{4\pi\varepsilon_o}\left\{\frac{(R - d\cos\theta)}{[R^2 + d^2 - 2Rd\cos\theta]^{3/2}} - \frac{(1 - \frac{R}{d}\cos\theta)}{Rd[1 + \frac{R^2}{d^2} - \frac{2R}{d}\cos\theta]^{3/2}}\right.$$

$$\left. + \frac{(1 + \frac{R}{d}\cos\theta)}{Rd[1 + \frac{R^2}{d^2} + \frac{2R}{d}\cos\theta]^{3/2}} - \frac{(R + d\cos\theta)}{[R^2 + d^2 + 2rd\cos\theta]^{3/2}}\right\}$$

The surface charge density is then

$$\sigma_f(r{=}R) = \varepsilon_o E_r(r{=}R)$$

with total charge

$$q_T(r{=}R) = 2\pi R^2 \int_{\theta=0}^{\pi/2} \sigma_f(r{=}R)\sin\theta\, d\theta = -2\pi R^2 \int_{u=1}^{0} \sigma_f(r{=}R)\, du \qquad (u = \cos\theta)$$

$$= \frac{qR^2}{2}\left\{\frac{-(Ru - d)}{R^2[R^2 + d^2 - 2Rdu]^{1/2}} + \frac{(Ru + d)}{R^2[R^2 + d^2 + 2Rdu]^{1/2}}\right.$$

$$\left.\left. + \frac{(u - \frac{R}{d})}{Rd[1 + \frac{R^2}{d^2} - \frac{2Ru}{d}]^{1/2}} - \frac{(u + \frac{R}{d})}{Rd[1 + \frac{R^2}{d^2} + \frac{2Ru}{d}]^{1/2}}\right\}\right|_{u=1}^{0}$$

$$= \frac{q}{2}\left\{\frac{2d}{[R^2 + d^2]^{1/2}} - \frac{2R^2}{d[R^2 + d^2]^{1/2}} + \frac{(R - d)}{(d - R)} - \frac{R + d}{(R + d)}\right.$$

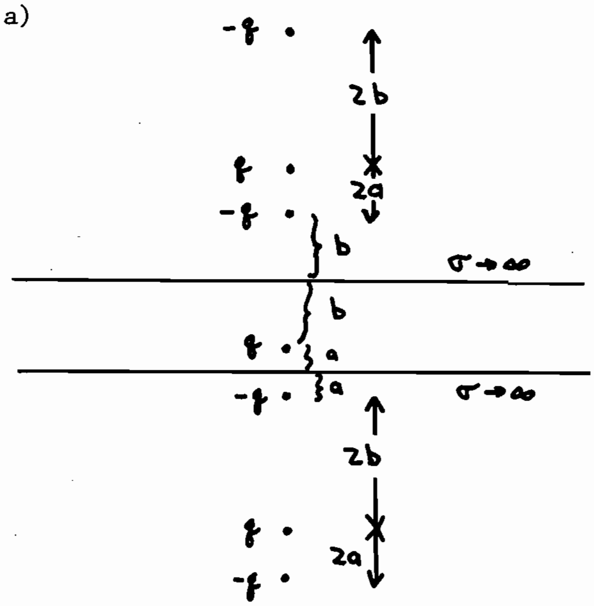
$$\left. - \frac{R(1 - \frac{R}{d})}{d(1 - \frac{R}{d})} + \frac{R(1 + \frac{R}{d})}{d(1 + \frac{R}{d})}\right\}$$

$$q_T(r{=}R) = q\left\{\frac{d - \frac{R^2}{d}}{[R^2 + d^2]^{1/2}} - 1\right\}$$

$$= -q\left\{1 + \frac{\frac{R^2}{d^2} - 1}{\sqrt{1 + \frac{R^2}{d^2}}}\right\}$$

which agrees with our earlier result.

43. a)



b) The total charge on each conductor is

$$q_T = -q[1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \ . \ . \ .] \text{ non-convergent series}$$

c) $q_{n+1} = -\dfrac{R_1}{b'_n} q'_n, \quad q'_{n+1} = -\dfrac{R_2}{b_n} q_n$

$$b_{n+1} = \frac{R_1^2}{b'_n}, \quad b'_{n+1} = \frac{R_2^2}{b_n}$$

d) $q_{n+1} = -\dfrac{R_1}{b'_n} q'_n = -\dfrac{R_1}{b'_n}\left(-\dfrac{R_2}{b_{n-1}} q_{n-1}\right) = \dfrac{R_1 R_2}{\underbrace{b'_n b_{n-1}}_{R_2^2}} q_{n-1}$

$$q_{n+1} = \frac{R_1}{R_2} q_{n-1}$$

$$b_{n+1} = \frac{R_1^2}{b'_n} = \frac{R_1^2}{R_2^2} b_{n-1}$$

e) $q_n = A\lambda^n$

$$A[\lambda^{n+1} - \frac{R_1}{R_2}\lambda^{n-1}] = A\lambda^{n-1}[\lambda^2 - \frac{R_1}{R_2}] = 0 \rightarrow \lambda = \pm \left(\frac{R_1}{R_2}\right)^{1/2}$$

$$b_n = B\alpha^n$$

$$0 = B[\alpha^{n+1} - (\frac{R_1}{R_2})^2 \alpha^{n-1}] = B\alpha^{n-1}[\alpha^2 - (\frac{R_1}{R_2})^2] \to \alpha = \pm \frac{R_1}{R_2}$$

f) $$q_n = A_1(\frac{R_1}{R_2})^{n/2} + A_2(-1)^n(\frac{R_1}{R_2})^{n/2} = (\frac{R_1}{R_2})^{n/2}[A_1 + (-1)^n A_2]$$

$$b_n = B_1(\frac{R_1}{R_2})^n + B_2(-1)^n(\frac{R_1}{R_2})^n = (\frac{R_1}{R_2})^n[B_1 + (-1)^n B_2]$$

$$\left.\begin{aligned} q_1 &= -\frac{qR_1}{R_o} = (A_1 - A_2)(\frac{R_1}{R_2})^{1/2} \\ q_2 &= \frac{qR_1}{R_2} = (A_1 + A_2)(\frac{R_1}{R_2}) \end{aligned}\right\} \to \begin{aligned} A_1 - A_2 &= -q(\frac{R_1R_2}{R_o^2})^{1/2} \\ A_1 + A_2 &= q \end{aligned}$$

$$q_n = \begin{cases} q(\frac{R_1}{R_2})^{n/2} & n \text{ even} \\ -q\,\frac{R_1}{R_o}(\frac{R_1}{R_2})^{(n-1)/2} & n \text{ odd} \end{cases}$$

$$b_1 = \frac{R_1^2}{R_o} = (B_1 - B_2)\frac{R_1}{R_2} \to B_1 - B_2 = \frac{R_1R_2}{R_o}$$

$$b_2 = \frac{R_1^2R_o}{R_2^2} = (B_1 + B_2)(\frac{R_1}{R_2})^2 \to B_1 + B_2 = R_o$$

$$b_n = \begin{cases} R_o(\frac{R_1}{R_2})^n & n \text{ even} \\ \frac{R_1R_2}{R_o}(\frac{R_1}{R_2})^n & n \text{ odd} \end{cases}$$

g) $$q_T(R_1) = q\sum_{n=1}^{\infty}[(\frac{R_1}{R_2})^n - \frac{R_1}{R_o}(\frac{R_1}{R_2})^{n-1}]$$

$$= -q\,\frac{(R_2 - R_o)}{R_o}\underbrace{\sum_{n=1}^{\infty}(\frac{R_1}{R_2})^n}_{R_1/(R_2-R_1)}$$

$$= \frac{-q\,R_1(R_2 - R_o)}{R_o(R_2 - R_1)}$$

h)
$$q'_{n+1} = \frac{-R_2 q_n}{b_n} = \begin{cases} \dfrac{-R_2 q \left(\frac{R_1}{R_2}\right)^{n/2}}{R_o \left(\frac{R_1}{R_2}\right)^n} = \dfrac{-qR_2}{R_o}\left(\frac{R_2}{R_1}\right)^{n/2} & n \text{ even} \\[2ex] \dfrac{+\frac{R_2 R_1}{R_o}\, q\left(\frac{R_1}{R_2}\right)^{(n-1)/2}}{\frac{R_1 R_2}{R_o}\left(\frac{R_1}{R_2}\right)^n} = q\left(\frac{R_2}{R_1}\right)^{(n+1)/2} & n \text{ odd} \end{cases}$$

$$b'_{n+1} = \frac{R_2^2}{b_n} = \begin{cases} \dfrac{R_2^2}{R_o}\left(\frac{R_2}{R_1}\right)^n & n \text{ even} \\[2ex] \dfrac{R_2 R_o}{R_1}\left(\frac{R_2}{R_1}\right)^n & n \text{ odd} \end{cases}$$

i) $$q_T(R_2) = \sum_{n=0}^{\infty} q'_{n+1} = \frac{-qR_2}{R_o}\sum_{n=0}^{\infty}\left(\frac{R_2}{R_1}\right)^{n/2}$$

However, since $R_2/R_1 > 1$ the series is non-convergent.

j) From Gauss's law $\oint_S \varepsilon_o \overline{E}\cdot\overline{dS} = Q_T$ enclosed. If the surface is taken inside the outer spherical conductor where $\overline{E} = 0$ we have that the total charge enclosed is zero.

$$q_T(R_2) + q_T(R_1) + q = 0 \rightarrow q_T(R_2) = -q - q_T(R_1) = -q\left[1 - \frac{R_1(R_2 - R_o)}{R_o(R_2 - R_1)}\right]$$
$$= -q\,\frac{R_2(R_o - R_1)}{R_o(R_2 - R_1)}$$

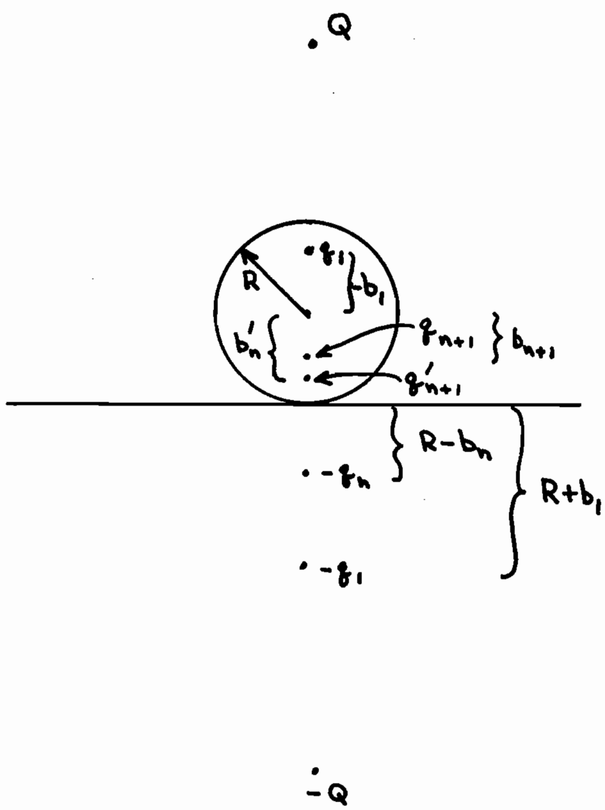
k) $d = R_2 - R_1$, $a = R_2 - R_o$, $b = R_o - R_1$

$$q_T(R_1) = \frac{-qR_1 a}{R_o d}, \quad q_T(R_2) = \frac{-qR_2}{R_o}\frac{b}{d}$$

$$\lim_{R_1/R_o \to 1} q_T(R_1) = \frac{-qa}{d}$$

$$R_2/R_o \to 1 \quad q_T(R_2) = \frac{-qb}{d}$$

44. a)



$$q_1 = -\frac{QR}{D-R}, \quad b_1 = \frac{-R^2}{D-R}$$

$$q_{n+1} = \frac{q_n R}{2R-b_n}, \quad b_{n+1} = \frac{R^2}{2R-b_n}$$

$$q'_{n+1} = \frac{q'_n R}{2R-b'_n}, \quad b'_{n+1} = \frac{R^2}{2R-b'_n}$$

$$q'_1 = \frac{QR}{D+R}, \quad b'_1 = \frac{R^2}{D+R}$$

$$q_{n+1} = \frac{q_n R}{2R-b_n}, \quad b_{n+1} = \frac{R^2}{2R-b_n}$$ (Charges induced by Q)

$\left(q_1 = -\frac{QR}{D-R}, \quad b_1 = \frac{-R^2}{D-R}\right)$

$$q'_{n+1} = \frac{q'_n R}{2R-b'_n}, \quad b'_{n+1} = \frac{R^2}{2R-b'_n}$$ [Charges induced by −Q (induced in plane by Q), $q'_1 = \frac{QR}{R+D}, \quad b'_1 = \frac{R^2}{D+R}$]

b) $$q_{n+1} = \frac{q_n R}{2R-b_n} = \frac{q_n}{R} b_{n+1} \rightarrow q_n = \frac{q_{n-1}}{R} b_n = q_{n-1}\left(2 - \frac{q_n}{q_{n+1}}\right)$$

$$\frac{q_n}{q_{n-1}} + \frac{q_n}{q_{n+1}} - 2 = 0 \rightarrow \frac{1}{q_{n+1}} - \frac{2}{q_n} + \frac{1}{q_{n-1}} = 0$$

c) Letting $P_n = \frac{1}{q_n}$

$$P_{n+1} - 2P_n + P_{n-1} = 0 \rightarrow P_n = A\lambda^n$$

$$A\lambda^{n-1}[\lambda^2 - 2\lambda + 1] = 0 \rightarrow \lambda = +1 \quad \text{(double root)}$$

d) $P_n = A_1 + A_2 n$

First set of image charges

$$\left.\begin{aligned} P_1 &= \frac{1}{q_1} = \frac{-(D-R)}{QR} = A_1 + A_2 \\ P_2 &= \frac{1}{q_2} = \frac{2R + \frac{R^2}{D-R}}{q_1 R} = A_1 + 2A_2 \end{aligned}\right\} \rightarrow \quad \begin{aligned} A_1 &= \frac{-R}{q_1(D-R)} = \frac{1}{Q} \\ A_2 &= \frac{D}{q_1(D-R)} = \frac{-D}{RQ} \end{aligned}$$

$$P_n = \frac{1}{Q}\left(1 - \frac{nD}{R}\right)$$

$$q_n = \frac{1}{P_n} = \frac{Q}{1 - \frac{nD}{R}}$$

Second set of image charges

$$P'_n = (A'_1 + nA'_2)$$

$$\left.\begin{aligned} P'_1 &= \frac{1}{q'_1} = \frac{D+R}{RQ} = (A'_1 + A'_2) \\ P'_2 &= \frac{1}{q'_2} = \frac{2R - \frac{R^2}{D+R}}{q'_1 R} = (A'_1 + 2A'_2) \end{aligned}\right\} \rightarrow \quad \begin{aligned} A'_1 &= \frac{R}{(D+R)q'_1} = \frac{1}{Q} \\ A'_2 &= \frac{1}{q'_1}\,\frac{D}{D+R} = \frac{D}{RQ} \end{aligned}$$

$$P'_n = \frac{1}{Q}\left[1 + \frac{nD}{R}\right]$$

$$q'_n = \frac{1}{P'_n} = \frac{Q}{\left[1 + \frac{nD}{R}\right]}$$

$$b_n = \frac{-q_n}{q_{n+1}} R + 2R$$

$$= \frac{-R\left[1 - \frac{(n+1)D}{R}\right]}{\left[1 - \frac{nD}{R}\right]} + 2R$$

$$= R\left[2 - \frac{\left(1 - \frac{(n+1)D}{R}\right)}{\left(1 - \frac{nD}{R}\right)}\right]$$

$$= \frac{R\left[1 - \frac{D}{R}(n-1)\right]}{\left(1 - \frac{nD}{R}\right)}$$

$$b'_n = \frac{-q'_n}{q'_{n+1}} R + 2R$$

$$= \frac{R\left(1 + \frac{D}{R}(n-1)\right)}{(1 + \frac{nD}{R})}$$

e) $$q_T = \sum_{n=1}^{\infty} (q_n + q'_n)$$

$$= Q \sum_{n=1}^{\infty} \left(\frac{1}{1 - \frac{nD}{R}} + \frac{1}{1 + \frac{nD}{R}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2Q}{1 - (\frac{D}{R})^2 n^2} = \sum_{n=1}^{\infty} \frac{A}{[1 - an^2]} \; ; \quad A = 2Q, \quad a = (\frac{D}{R})^2$$

f) $$\lim_{D\to\infty} E_z = \frac{-2Q}{4\pi\varepsilon_o D^2} = -E_o \to \frac{Q}{D^2} = 2\pi\varepsilon_o E_o$$

g) $$\lim_{D\to\infty} \quad q_T = -\sum_{n=1}^{\infty} \frac{2Q}{D^2}\frac{R^2}{n^2} = -4\pi\varepsilon_o E_o R^2 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\pi^2/6}$$

$$\frac{Q}{D^2} = 4\pi\varepsilon_o E_o$$

$$q_T = -4\pi\varepsilon_o E_o R^2 \frac{\pi^2}{6}$$

45. a) Because the image charge for a conducting sphere is less than the inducing charge it is impossible to solve this problem with only two image charges as we were able to do for parallel cylinders in (2.6). It will now require an infinite set of image charges to meet the boundary conditions.

To make the sphere an equipotential surface, place a point charge q_1 at the center of the sphere. To keep the plane interface at zero potential an image charge $-q_1$ must appear symmetrically a distance D behind the interface. However, this charge now results in an image charge $q_2 = q_1R/2D$ at a position $b_2 = R^2/2D$ from the sphere center. This charge q_2 must then have an image charge $-q_2$ a distance $D - b_2$ back from the plane interface. Then again, an image charge $q_3 = q_2R/(2D - b_2)$ a distance $b_3 = R^2/(2D - b_2)$ is necessary to keep the sphere at constant potential. This process continues on for an infinite number of positive image charges confined within the sphere and negative charges within the plane conductor. Each successive image charge gets smaller in magnitude and closer to the sphere surface and closer to the plane boundary. The nth image charge is related to the previous one as

$$q_n = \frac{q_{n-1} R}{2D - b_{n-1}}, \quad b_n = \frac{R^2}{2D - b_{n-1}}$$

$$q_1 = 4\pi\varepsilon_o RV_o, \quad q_2 = \frac{q_1 R}{2D} = \frac{4\pi\varepsilon_o R^2 V_o}{2D}$$

b) $$\frac{q_n}{q_{n-1}} = \frac{R}{2D - b_{n-1}} = \frac{b_n}{R} \rightarrow b_n = \frac{R\, q_n}{q_{n-1}}$$

Letting $n \rightarrow n + 1$

$$\frac{q_{n+1}}{q_n} = \frac{R}{2D - b_n} \rightarrow \frac{q_n}{q_{n+1}} = \frac{2D - b_n}{R} = \frac{2D - Rq_n/q_{n-1}}{R}$$

$$\frac{q_n}{q_{n+1}} + \frac{q_n}{q_{n-1}} = \frac{2D}{R} \equiv C$$

c) Letting $P_n = \frac{1}{q_n}$

$$\frac{1}{q_{n+1}} + \frac{1}{q_{n-1}} - \frac{C}{q_n} = 0 \rightarrow P_{n+1} - CP_n + P_{n-1} = 0$$

Let $P_n = A\lambda^n$

$$A\lambda^{n-1}[\lambda^2 - C\lambda + 1] = 0 \rightarrow \lambda^2 - C\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{C}{2} \pm \sqrt{(\frac{C}{2})^2 - 1} \rightarrow \lambda_2 = \frac{1}{\lambda_1}$$

$$P_n = A_1\lambda^n + A_2\lambda^{-n}$$

$$\left.\begin{aligned} P_1 &= \frac{1}{q_1} = A_1\lambda + \frac{A_2}{\lambda} \\ P_2 &= \frac{1}{q_2} = \frac{C}{q_1} = A_1\lambda^2 + \frac{A_2}{\lambda^2} \end{aligned}\right\} \rightarrow A_1 = -A_2 = \frac{\lambda}{q_1(\lambda^2 - 1)}$$

$$q_n = \frac{1}{P_n} = \frac{\lambda^n}{A_1\lambda^{2n} + A_2} = \frac{q_1(\lambda^2 - 1)}{\lambda} \frac{\lambda^n}{(\lambda^{2n} - 1)}$$

d) $$b_n = 2d - \frac{q_n R}{q_{n+1}} = 2d - \frac{R(\lambda^{2(n+1)} - 1)}{\lambda(\lambda^{2n} - 1)}$$

$$\lim_{n\to\infty} b_n = d\{1 - [1 - (\frac{R}{d})^2]^{1/2}\}$$

e) $$C = \frac{q_1 + q_2 + q_3 + \ldots}{V_o} = \frac{\sum_{n=1}^{\infty} q_n}{V_o}$$

$$= \frac{4\pi\varepsilon R}{q_1} \sum_{n=1}^{\infty} q_n$$

$$= \frac{4\pi\varepsilon R(\lambda^2 - 1)}{\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{\lambda^{2n} - 1}$$

$$= 4\pi\varepsilon R(\lambda^2 - 1)\left\{\frac{1}{\lambda^2 - 1} + \frac{\lambda}{\lambda^4 - 1} + \frac{\lambda^2}{\lambda^6 - 1} + \frac{\lambda^3}{\lambda^8 - 1} + \ldots\right\}$$

$\rightarrow C_o = 4\pi\varepsilon R$ (Capacitance of an isolated sphere)

$$\lambda = \frac{c}{2} - [(\frac{c}{2})^2 - 1]^{1/2}, \quad c = \frac{2D}{R}$$

f) $$q'_n = \frac{-R_2 q_n}{D \mp b_n}, \quad b'_n = \frac{R_2^2}{D \mp b_n}$$ (Upper signs for adjacent spheres, lower signs for smaller sphere R_1 inside larger one)

$$q_{n+1} = \mp \frac{q'_n R_1}{D - b'_n} = \pm \frac{R_1 R_2 q_n}{D(D \mp b_n) - R_2^2}$$

$$b_{n+1} = \pm \frac{R_1^2}{D - b'_n} = \pm \frac{R_1^2(D \mp b_n)}{D(D \mp b_n) - R_2^2} = \frac{R_1}{R_2} \frac{q_{n+1}}{q_n} (D \mp b_n)$$

g) Eliminating $(D - b_n)$ in the unprimed variables

$$\frac{q_n}{q_{n+1}} = \pm \frac{D}{R_1^2} \frac{q_n}{q_{n+1}} b_{n+1} \mp \frac{R_2}{R_1}$$

Dropping the index by 1

$$\frac{q_{n-1}}{q_n} = \pm \frac{D}{R_1^2} \frac{q_{n-1}}{q_n} b_n \mp \frac{R_2}{R_1}$$

Since

$$b_n = -\frac{1}{D}\left[\pm \frac{R_1 R_2 q_n}{q_{n+1}} + R_2^2 - D^2\right]$$

we have

$$\frac{1}{q_{n+1}} + \frac{1}{q_{n-1}} \mp \frac{C}{q_n} = 0; \quad C = \frac{D^2 - R_1^2 - R_2^2}{R_1R_2}$$

Letting $P_n = \frac{1}{q_n}$

$$P_{n+1} \mp C\,P_n + P_{n-1} = 0 \rightarrow P_n = \frac{1}{q_n} = A_1\lambda^n + A_2\lambda^{-n}$$

h) $\lambda = \pm\frac{C}{2} - [(\frac{C}{2})^2 - 1]^{1/2}$

$$\frac{1}{q_1} = A_1\lambda + \frac{A_2}{\lambda}$$

$$\frac{1}{q_2} = \pm\frac{(D^2 - R_2^2)}{R_1R_2q_1} = \pm\frac{1}{q_1}\left(C + \frac{R_1}{R_2}\right) = A_1\lambda^2 + \frac{A_2}{\lambda^2}$$

$$A_1 = \mp\frac{A_2\left(\lambda \pm \frac{R_1}{R_2}\right)}{\lambda\left(\pm 1 + \frac{R_1\lambda}{R_2}\right)} = -A_2\left[\frac{(\lambda R_2 \pm R_1)}{D\lambda}\right]^2 = -A_2\xi^2$$

i) $q_n = \frac{q_1(1 - \xi^2)}{\lambda}\,\frac{\lambda^n}{1 - \xi^2\lambda^{2(n-1)}}$

$$C = \frac{1}{V_o}\sum_{n=1}^{\infty} q_n = \frac{4\pi\varepsilon_o R_1}{q_1}\sum_{n=1}^{\infty} q_n$$

$$= \frac{4\pi\varepsilon_o R_1}{\lambda}(1 - \xi^2)\sum_{n=1}^{\infty}\frac{\lambda^n}{(1 - \xi^2\lambda^{2(n-1)})}$$

$$= 4\pi\varepsilon_o R_1(1 - \xi^2)\left\{\frac{1}{1 - \xi^2} + \frac{\lambda}{1 - \xi^2\lambda^2} + \frac{\lambda^2}{1 - \xi^4\lambda^4} + \ldots\right\}$$

j) When $D = 0 \rightarrow \lambda = \frac{R_1}{R_2} \rightarrow \xi = 0$

$$C = 4\pi\varepsilon_o R_1\{1 + \lambda + \lambda^2 + \lambda^3 + \ldots\}$$

$$= \frac{4\pi\varepsilon_o R_1}{1 - \lambda} = \frac{4\pi\varepsilon_o R_1}{1 - \frac{R_1}{R_2}} = \frac{4\pi\varepsilon_o R_1R_2}{R_2 - R_1}$$

CHAPTER 3

POLARIZATION AND CONDUCTION

Section 3.1

1. a) $V = \frac{-\lambda}{2\pi\varepsilon_o} \ln \frac{r_1}{r_2}$; $r_1 = [r^2 + (\frac{d}{2})^2 - rd\sin\phi]^{1/2}$

$$r_2 = [r^2 + (\frac{d}{2})^2 + rd\sin\phi]^{1/2}$$

b) For $r >> d$

$$r_1 \approx r(1 - \frac{d}{2r}\sin\phi),\quad r_2 \approx r(1 + \frac{d}{2r}\sin\phi)$$

$$V \approx \frac{-\lambda}{2\pi\varepsilon_o}[\ln(1 - \frac{d}{2r}\sin\phi) - \ln(1 + \frac{d}{2r}\sin\phi)]$$

$$\approx \frac{\lambda d\sin\phi}{2\pi\varepsilon_o r}$$

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\bar{i}_r - \frac{1}{r}\frac{\partial V}{\partial\phi}\bar{i}_\phi$$

$$= \frac{\lambda d}{2\pi\varepsilon_o r^2}[\sin\phi\,\bar{i}_r - \cos\phi\,\bar{i}_\phi]$$

Let $p = \lambda d$ (per unit length).

c) $\frac{dr}{rd\phi} = \frac{E_r}{E_\phi} = -\tan\phi \rightarrow \ln r = \ln\cos\phi + \text{constant}$

$$\frac{r}{\cos\phi} = \text{constant}$$

2. a) $\bar{p} = \int \bar{r}dq$

$$p_z = \int_{-L}^{0} -\lambda_o z dz + \int_{0}^{L} \lambda_o z dz = -\frac{\lambda_o z^2}{2}\Big|_{-L}^{0} + \frac{\lambda_o z^2}{2}\Big|_{0}^{L} = \lambda_o L^2$$

b) $p_z = \int_{-L}^{0} -\lambda_o(1 + \frac{z}{L})zdz + \int_{0}^{L} \lambda_o(1 - \frac{z}{L})zdz$

$$= -\lambda_o\left(\frac{z^2}{2} + \frac{z^3}{3L}\right)\Big|_{-L}^{0} + \lambda_o\left(\frac{z^2}{2} - \frac{z^3}{3L}\right)\Big|_{0}^{L}$$

$$= \lambda_o L^2 - \frac{2\lambda_o L^2}{3} = \frac{1}{3}\lambda_o L^2$$

c) $$\left.\begin{aligned} p_z &= \int_0^L \lambda_o z\,dz = \frac{\lambda_o L^2}{2} \\ p_y &= \int_0^L -\lambda_o y\,dy = -\frac{\lambda_o L^2}{2} \end{aligned}\right\} \rightarrow \overline{p} = p_y\overline{i}_y + p_z\overline{i}_z = \frac{\lambda_o L^2}{2}(\overline{i}_z - \overline{i}_y)$$

d) $$p_z = \int_{d/2}^{(d/2)+L} \lambda_o z\,dz - \int_{-L-(d/2)}^{-d/2} \lambda_o z\,dz$$

$$= \frac{\lambda_o z^2}{2}\Big|_{d/2}^{(d/2)+L} - \frac{\lambda_o z^2}{2}\Big|_{-(L+\frac{d}{2})}^{-d/2}$$

$$= \lambda_o\left[\left(\frac{d}{2} + L\right)^2 - \left(\frac{d}{2}\right)^2\right]$$

$$= \lambda_o L(L + d)$$

e) $$\overline{p} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} R\overline{i}_r \frac{Q}{2\pi} \sin\theta\,d\theta\,d\phi - \int_{\phi=0}^{2\pi}\int_{\theta=\frac{\pi}{2}}^{\pi} R\overline{i}_r \frac{Q}{2\pi} \sin\theta\,d\theta\,d\phi$$

$$\overline{i}_r = \sin\theta\cos\phi\,\overline{i}_x + \sin\theta\sin\phi\,\overline{i}_y + \cos\theta\,\overline{i}_z$$

$$p_x = p_y = 0$$

$$p_z = QR\left[\int_{\theta=0}^{\pi/2} \sin\theta\cos\theta\,d\theta - \int_{\theta=\frac{\pi}{2}}^{\pi} \sin\theta\cos\theta\,d\theta\right]$$

$$= QR\left[\frac{\sin^2\theta}{2}\Big|_{\theta=0}^{\pi/2} - \frac{\sin^2\theta}{2}\Big|_{\theta=\frac{\pi}{2}}^{\pi}\right]$$

$$= QR$$

f) $$dp_z = \frac{Q\,dr}{\frac{2}{3}\pi R^3}\,2\pi r^3 = \frac{3Qr^3 dr}{R^3}$$

$$p_z = \frac{3Q}{R^3}\int_0^R r^3 dr$$

$$= \frac{3}{4}QR$$

3. $$V = \frac{1}{4\pi\varepsilon_o}\left[-\frac{Q}{r_1} + \frac{2Q}{r} - \frac{Q}{r_2}\right]$$

$$\frac{1}{r_1} = \frac{1}{[r^2 + d^2 - 2rd\cos\theta]^{1/2}} = \frac{1}{r[1 + \frac{d}{r}[(\frac{d}{r}) - 2\cos\theta]]^{1/2}}$$

$$\approx \frac{1}{r}\{1 - \frac{1}{2}(\frac{d}{r})[(\frac{d}{r}) - 2\cos\theta] - \frac{3}{4}(\frac{d}{r})^2[(\frac{d}{r}) - 2\cos\theta]^2\}$$

$$\approx \frac{1}{r}\{1 + (\frac{d}{r})\cos\theta - \frac{1}{2}(\frac{d}{r})^2[1 - 3\cos^2\theta]\}$$

Similarly

$$\frac{1}{r_2} = \frac{1}{[r^2 + d^2 + 2rd\cos\theta]^{1/2}} = \frac{1}{r[1 + \frac{d}{r}[(\frac{d}{r}) + 2\cos\theta]]^{1/2}}$$

$$\approx \frac{1}{r}\{1 - \frac{d}{r}\cos\theta - \frac{1}{2}(\frac{d}{r})^2[1 - 3\cos^2\theta]\}$$

Then

$$V \approx \frac{Q}{4\pi\varepsilon_o r}[(\frac{d}{r})^2(1 - 3\cos^2\theta)]$$

$$\approx \frac{Qd^2}{4\pi\varepsilon_o r^3}(1 - 3\cos^2\theta)$$

$$\overline{E} = -\nabla V = -\frac{\partial V}{\partial r}\overline{i}_r - \frac{1}{r}\frac{\partial V}{\partial\theta}\overline{i}_\theta$$

$$= \frac{Qd^3}{4\pi\varepsilon_o r^4}[3(1 - 3\cos^2\theta)\overline{i}_r - 6\cos\theta\sin\theta\overline{i}_\theta]$$

4. a) $\rho = -\rho_o(1 - \frac{r}{R_o}) \qquad r < R_o$

$$-Q = -4\pi \int_0^{R_o} \rho_o(1 - \frac{r}{R_o}) r^2 dr$$

$$= -4\pi\rho_o[\frac{R_o^3}{3} - \frac{R_o^4}{4R_o}]$$

$$= -\frac{\pi\rho_o R_o^3}{3} \rightarrow \rho_o = \frac{3Q}{\pi R_o^3}$$

b) The electric field due to the volume charge distribution is

$$\oint_S \varepsilon_o \overline{E}\cdot\overline{dS} = \varepsilon_o E_r 4\pi r^2 = -4\pi\rho_o \int_0^r (1 - \frac{r}{R_o}) r^2 dr$$

$$= -4\pi\rho_o[\frac{r^3}{3} - \frac{r^4}{4R_o}]$$

$$E_r = -\frac{\rho_o}{\varepsilon_o}[\frac{r}{3} - \frac{r^2}{4R_o}] \qquad r < R_o$$

$$\overline{F} = Q[E_{LOC} - \frac{\rho_o}{\varepsilon_o}(\frac{d}{3} - \frac{d^2}{4R_o})] = 0$$

$$d^2 - \frac{4R_o d}{3} + \frac{4R_o\varepsilon_o E_{LOC}}{\rho_o} = 0$$

$$d = \frac{2}{3}R_o - \sqrt{\frac{4}{9}R_o^2 - \frac{4R_o\varepsilon_o E_{LOC}}{\rho_o}}$$

$$= \frac{2}{3}R_o [1 - \sqrt{1 - \frac{9\varepsilon_o E_{LOC}}{\rho_o R_o}}]$$

c) $\frac{9\varepsilon_o E_{LOC}}{\rho_o R} << 1$

$$d \approx \frac{2}{3}R_o [\frac{9}{2}\frac{\varepsilon_o E_{LOC}}{\rho_o R_o}]$$

$$= \frac{3\varepsilon_o E_{LOC}}{\rho_o}$$

$$\overline{p} = \alpha\overline{E}_{LOC} = Q\overline{d} \rightarrow \alpha = \frac{3\varepsilon_o Q}{\rho_o} = \varepsilon_o \pi R_o^3$$

5. a) $$\overline{E}_T = E_o\overline{i}_z + \frac{2\overline{p}}{4\pi\varepsilon_o a^3}$$

$$= E_o\overline{i}_z + \frac{2\alpha\overline{E}_T}{4\pi\varepsilon_o a^3} \rightarrow \overline{E}_T = \frac{E_o\overline{i}_z}{[1 - \frac{\alpha}{2\pi\varepsilon_o a^3}]}$$

b) $$\overline{E}_T = E_o\overline{i}_z + \frac{2\alpha\overline{E}_T}{4\pi\varepsilon_o a^3}\sum_{n=1}^{\infty}\frac{2}{n^3}$$

$$\approx E_o\overline{i}_z + \frac{1.2\alpha\overline{E}_T}{\pi\varepsilon_o a^3} \rightarrow \overline{E}_T = \frac{E_o\overline{i}_z}{[1 - \frac{1.2\alpha}{\pi\varepsilon_o a^3}]}$$

c) $$\overline{p} = \alpha\overline{E}_T = \frac{\alpha E_o\overline{i}_z}{[1 - \frac{1.2\alpha}{\pi\varepsilon_o a^3}]}$$

$$\overline{P} = N\overline{p} = \frac{\overline{p}}{a^3} = \frac{\alpha E_o\overline{i}_z}{a^3[1 - \frac{1.2\alpha}{\pi\varepsilon_o a^3}]} = \chi_e\varepsilon_o\overline{E}_o$$

$$\chi_e = \frac{\alpha}{\varepsilon_o a^3[1 - \frac{1.2\alpha}{\pi\varepsilon_o a^3}]} \rightarrow \varepsilon = \varepsilon_o(1 + \chi_e)$$

6. a) The electric field within the sphere due to the volume charge distribution is

$$\overline{E}_- = \frac{-Qr}{4\pi\varepsilon_o R_o^3}\overline{i}_r$$

The electric field due to the point charge with reference to a spherical coordinate system with origin at the center of the sphere is

$$\overline{E}_+ = \frac{Q\,\overline{i}_{QP}}{4\pi\varepsilon_o r_{QP}^2} = \frac{Q(r\overline{i}_r - d\overline{i}_z)}{4\pi\varepsilon_o[r^2 + d^2 - 2rd\cos\theta]^{3/2}}$$

The total dipole electric field is then

$$\overline{E}_{dip} = \overline{E}_- + \overline{E}_+ = \frac{-Qr\overline{i}_r}{4\pi\varepsilon_o R_o^3} + \frac{Q(r\overline{i}_r - d\overline{i}_z)}{4\pi\varepsilon_o[r^2 + d^2 - 2rd\cos\theta]^{3/2}}$$

b) $\bar{i}_r = \sin\theta\cos\phi\,\bar{i}_x + \sin\theta\sin\phi\,\bar{i}_y + \cos\theta\,\bar{i}_z$

$$\langle E_x\rangle_{dip} = \frac{1}{\frac{4}{3}\pi R_o^3}\int_{r=0}^{R_o}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} E_{x_{dip}}\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

Since $E_{x_{dip}}$ is proportional to $\cos\phi$ which integrates to zero over ϕ

$$\int_{\phi=0}^{2\pi}\cos\phi = 0$$

the average x component of field is zero. Similarly E_y is proportional to $\sin\phi$ which also integrates to zero over ϕ.

c) $$E_{z_{dip}} = -\frac{Qr\cos\theta}{4\pi\varepsilon_o R^3} + \frac{Q(r\cos\theta - d)}{4\pi\varepsilon_o[r^2 + d^2 - 2rd\cos\theta]^{3/2}}$$

so that after integrating over ϕ

$$\langle E_z\rangle_{dip} = \frac{2\pi}{\frac{4}{3}\pi R_o^3}\int_{r=0}^{R_o}\int_{\theta=0}^{\pi}\frac{Q}{4\pi\varepsilon_o}\left[-\frac{r\cos\theta}{R_o^3} + \frac{(r\cos\theta - d)}{[r^2 + d^2 - 2rd\cos\theta]^{3/2}}\right]r^2\sin\theta\,dr\,d\theta$$

The first term in the integrand integrates to zero over θ. The second term is integrated by introducing the change of variable

$$u = r^2 + d^2 - 2rd\cos\theta,\quad du = 2rd\sin\theta\,d\theta$$

so that

$$\langle E_z\rangle_{dip} = \frac{3Q}{8\pi\varepsilon_o R_o^3}\int_{r=0}^{R_o}\int_{u=(r-d)^2}^{(r+d)^2}\frac{r(r\cos\theta - d)}{2du^{3/2}}\,dr\,du$$

Now

$$r\cos\theta = -\frac{(u - r^2 - d^2)}{2d}$$

so that the integral reduces to

$$\langle E_z\rangle_{dip} = \frac{3Q}{32\pi\varepsilon_o R_o^3 d^2}\int_{r=0}^{R_o}\int_{u=(r-d)^2}^{(r+d)^2}\left[\frac{-r}{u^{1/2}} + \frac{r(r^2 - d^2)}{u^{3/2}}\right]dr\,du$$

$$= \frac{3Q}{32\pi\varepsilon_o R_o^3 d^2}\int_{r=0}^{R_o}\left[-2ru^{1/2} - \frac{2r(r^2 - d^2)}{u^{1/2}}\right]\Bigg|_{u=(r-d)^2}^{(r+d)^2} dr$$

$$= \frac{-3Q}{16\pi\varepsilon_o R_o^3 d^2} \int_{r=0}^{R_o} r\left[r + d - |r - d| + r - d - \frac{(r + d)(r - d)}{|r - d|}\right] dr$$

When r is greater than d the integrand is zero so that the integral reduces to

$$\langle E_z \rangle_{dip} = \frac{-3Q}{4\pi\varepsilon_o R_o^3 d^2} \int_{r=0}^{d} r^2 dr$$

$$= \frac{-Qd}{4\pi\varepsilon_o R_o^3} = \frac{-p_z}{4\pi\varepsilon_o R_o^3}$$

d) $$\langle \overline{E} \rangle_{dip} = \frac{-\overline{p}}{4\pi\varepsilon_o R_o^3} = \frac{-\overline{P}}{3\varepsilon_o}$$

$$\overline{P} = N\alpha\overline{E}_{LOC} = N\alpha[\overline{E} - \langle\overline{E}\rangle_{dip}] = N\alpha[\overline{E} + \frac{\overline{P}}{3\varepsilon_o}]$$

$$\overline{P} = \frac{N\alpha}{1 - \frac{N\alpha}{3\varepsilon_o}} \overline{E}$$

7. a) $$\overline{d} = \frac{4\pi\varepsilon_o R_o^3}{Q} \overline{E}_o$$

b) $$m \frac{d^2\overline{d}}{dt^2} = Q[\overline{E}_o - \frac{Q\overline{d}}{4\pi\varepsilon_o R_o^3}]$$

If $\overline{E}_o = 0$

$$\frac{d^2\overline{d}}{dt^2} + \frac{Q^2}{4\pi\varepsilon_o R_o^3 m} \overline{d} = 0$$

$$\overline{d} = \overline{d}_o \cos\omega_o t; \quad \overline{d}_o = \frac{4\pi\varepsilon_o R_o^3}{Q} \overline{E}_o, \quad \omega_o = \sqrt{\frac{Q^2}{4\pi\varepsilon_o R_o^3 m}}$$

c) $Q = e = 1.6 \times 10^{-19}$ Coulombs, $R_o = 10^{-10}$ meter, $m = 9.1 \times 10^{-31}$ kg.

$\omega_o = 1.6 \times 10^{16}$ rad/sec $\rightarrow f_o = \frac{\omega_o}{2\pi} = 2.5 \times 10^{15}$ Hz

d) $$m \frac{d^2\overline{d}}{dt^2} = -\beta \frac{d\overline{d}}{dt} - \frac{Q^2}{4\pi\varepsilon_o R_o^3} \overline{d} + Q\mathrm{Re}(\overline{E}_o e^{j\omega t})$$

$$\frac{d^2\overline{d}}{dt^2} + \gamma \frac{d\overline{d}}{dt} + \omega_o^2 \overline{d} = \frac{Q}{m} \mathrm{Re}(\overline{E}_o e^{j\omega t}); \quad \gamma = \frac{\beta}{m}$$

e) $\overline{d} = \mathrm{Re}\ \hat{d}\ e^{j\omega t}$

$$\hat{d} = \frac{QE_o/m}{[-\omega^2 + \gamma j\omega + \omega_o^2]}$$

$$\hat{p} = Q\hat{d} = \frac{Q^2 E_o/m}{[-\omega^2 + \gamma j\omega + \omega_o^2]} = \hat{\alpha}\ E_o \rightarrow \hat{\alpha} = \frac{Q^2/m}{[-\omega^2 + \gamma j\omega + \omega_o^2]}$$

f) $$\hat{\varepsilon} = \varepsilon_o \left[1 + \frac{N\hat{\alpha}/\varepsilon_o}{1 - \frac{N\hat{\alpha}}{3\varepsilon_o}}\right]$$

$$= \varepsilon_o \left[1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2 + \gamma j\omega - \frac{\omega_p^2}{3}}\right]; \quad \omega_p^2 = \frac{Q^2 N}{m\varepsilon_o}$$

$$= \varepsilon_o \left\{1 + \frac{\omega_p^2[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3}) - \gamma j\omega]}{[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})^2 + \gamma^2\omega^2]}\right\}$$

$$\varepsilon_r = \varepsilon_o \left\{1 + \frac{\omega_p^2(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})}{[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})^2 + \gamma^2\omega^2]}\right\}$$

$$\varepsilon_i = \frac{-\varepsilon_o \omega_p^2 \gamma\omega}{[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})^2 + \gamma^2\omega^2]}$$

g) If the lossy dielectric is placed between parallel plate electrodes

$$\frac{\hat{i}}{A} = \frac{j\omega\hat{\varepsilon}\hat{v}}{s} \rightarrow Y = \frac{\hat{i}}{\hat{v}} = \frac{j\omega\hat{\varepsilon}A}{s}$$

$$= \frac{j\omega\varepsilon_o A}{s} \left\{1 + \frac{\omega_p^2[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3}) - \gamma j\omega]}{[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})^2 + \gamma^2\omega^2]}\right\}$$

$$= \frac{j\omega\varepsilon_o A}{s}\left\{1 + \frac{\dfrac{\omega_p^2}{(\omega_o^2 - \frac{\omega_p^2}{3})^2}\left[(\omega_o^2 - \omega^2 - \frac{\omega_p^2}{3})^2 - \gamma j\omega\right]}{\left[\left[1 - \dfrac{\omega^2}{(\omega_o^2 - \frac{\omega_p^2}{3})}\right]^2 + \dfrac{\gamma^2\omega^2}{(\omega_o^2 - \frac{\omega_p^2}{3})^2}\right]}\right\}$$

The admittance of the equivalent circuit shown is

$$Y = j\omega C_1\left\{1 + \frac{\frac{C_2}{C_1}\left[1 - LC_2\omega^2 - RC_2 j\omega\right]}{\left[(1 - LC_2\omega^2)^2 + (RC_2\omega)^2\right]}\right\}$$

so that

$$C_1 = \frac{\varepsilon_o A}{s},\ LC_2 = \frac{1}{(\omega_o^2 - \frac{\omega_p^2}{3})},\ RC_2 = \frac{\gamma}{(\omega_o^2 - \frac{\omega_p^2}{3})},\ \frac{C_2}{C_1} = \frac{\omega_p^2}{(\omega_o^2 - \frac{\omega_p^2}{3})}$$

$$\rightarrow C_2 = \frac{\varepsilon_o A}{s}\frac{\omega_p^2}{(\omega_o^2 - \frac{\omega_p^2}{3})},\ L = \frac{1}{\omega_p^2 \frac{\varepsilon_o A}{s}},\ R = \frac{\gamma}{\omega_p^2 \frac{\varepsilon_o A}{s}}$$

h) When $m \rightarrow 0$, ω_p^2, ω_o^2, and $\gamma \rightarrow \infty$ so that

$$\varepsilon_r = \varepsilon_o\left[1 + \frac{\omega_p^2(\omega_o^2 - \frac{\omega_p^2}{3})}{(\omega_o^2 - \frac{\omega_p^2}{3})^2 + \gamma^2\omega^2}\right]$$

$\lim\limits_{m\rightarrow 0}$

$$\varepsilon_i = \frac{-\varepsilon_o\,\omega_p^2\,\gamma\omega}{\left[(\omega_o^2 - \frac{\omega_p^2}{3})^2 + \gamma^2\omega^2\right]}$$

From ε_r we have

$$\left[(\omega_o^2 - \frac{\omega_p^2}{3})^2 + \gamma^2\omega^2\right] = \frac{\omega_p^2(\omega_o^2 - \frac{\omega_p^2}{3})\varepsilon_o}{\varepsilon_r - \varepsilon_o}$$

so that

$$\varepsilon_i = \frac{-\gamma\omega(\varepsilon_r - \varepsilon_o)}{(\omega_o^2 - \frac{\omega_p^2}{3})}$$

Squaring both sides and eliminating $\gamma^2\omega^2$ yields

$$\varepsilon_i^2 + (\varepsilon_r - \varepsilon_o)^2 - \frac{(\varepsilon_r - \varepsilon_o)\varepsilon_o\omega_p^2}{(\omega_o^2 - \frac{\omega_p^2}{3})} = 0$$

By completing the square this relation becomes

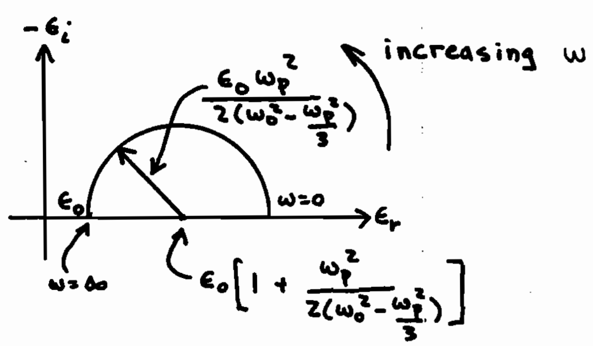
$$\varepsilon_i^2 + \left[\varepsilon_r - \varepsilon_o\left(1 + \frac{\omega_p^2}{2(\omega_o^2 - \frac{\omega_p^2}{3})}\right)\right]^2 = \left[\frac{\varepsilon_o\omega_p^2}{2(\omega_o^2 - \frac{\omega_p^2}{3})}\right]^2$$

which is the equation of a circle with radius

$$r = \frac{\varepsilon_o\omega_p^2}{2(\omega_o^2 - \frac{\omega_p^2}{3})}$$

and center at

$$\varepsilon_i = 0, \quad \varepsilon_r = \varepsilon_o\left[1 + \frac{\omega_p^2}{2(\omega_o^2 - \frac{\omega_p^2}{3})}\right]$$



When m = 0, the equivalent circuit of (g) has no inductance.

i) To find $\varepsilon_{i_{max}}$

$$\frac{\partial \varepsilon_i}{\partial \omega} = \frac{-\varepsilon_o \omega_p^2 \gamma}{\left[\left(\omega_o^2 - \frac{\omega_p^2}{3}\right)^2 + \gamma^2\omega^2\right]^2}\left[-2\gamma^2\omega^2 + \left(\omega_o^2 - \frac{\omega_p^2}{3}\right)^2 + \gamma^2\omega^2\right] = 0$$

which is zero at frequency

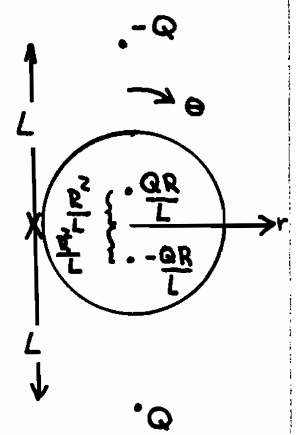
$$\omega_{max}^2 = \frac{\left(\omega_o^2 - \frac{\omega_p^2}{3}\right)^2}{\gamma^2}$$

so that

$$(\varepsilon_i)_{max} = \frac{\varepsilon_o \omega_p^2 \gamma \omega_{max}}{2(\gamma^2 \omega_{max}^2)} = \frac{\varepsilon_o \omega_p^2}{2\gamma\omega_{max}} = \frac{\varepsilon_o \omega_p^2}{2\left(\omega_o^2 - \frac{\omega_p^2}{3}\right)}$$

8. a) Each charge Q induces image charges in the sphere $-\frac{QR}{L}$. Along z axis

$$E_z = \frac{Q}{4\pi\varepsilon_o}\left[\frac{(z+L)}{|z+L|^3} - \frac{(z-L)}{|z-L|^3} - \frac{\left(z+\frac{R^2}{L}\right)}{\left|z+\frac{R^2}{L}\right|^3} + \frac{\left(z-\frac{R^2}{L}\right)}{\left|z-\frac{R^2}{L}\right|^3}\right]$$



In z=0 plane

$$E_z = \frac{Q}{4\pi\varepsilon_o}\left[\frac{2L}{[L^2 + r^2]^{3/2}} - \frac{2R^3}{L^2\left[\left(\frac{R^2}{L}\right)^2 + r^2\right]^{3/2}}\right]$$

b) $\lim_{L>>r}$ $\quad E_z \approx \frac{Q}{2\pi\varepsilon_o}\left[\frac{1}{L^2}\right], \quad \frac{Q}{L^2} = 2\pi\varepsilon_o E_o$

c) In this limit, the two image charges form a point dipole with moment

$$p_z = \frac{QR}{L}\frac{2R^2}{L} = \frac{2QR^3}{L^2} = 4\pi\varepsilon_o R^3 E_o$$

The electric field is then the superposition of the imposed field $E_o\bar{i}_z$ with the induced dipole field

$$\bar{E}_T = E_o\bar{i}_z + \frac{p_z}{4\pi\varepsilon_o r^3}[2\cos\theta\,\bar{i}_r + \sin\theta\,\bar{i}_\theta]$$

$$= E_o[\bar{i}_r\cos\theta - \bar{i}_\theta\sin\theta] + \frac{E_o R^3}{r^3}[2\cos\theta\,\bar{i}_r + \sin\theta\,\bar{i}_\theta]$$

$$= E_o\left\{\left(1 + \frac{2R^3}{r^3}\right)\cos\theta\,\bar{i}_r - \left(1 - \frac{R^3}{r^3}\right)\sin\theta\,\bar{i}_\theta\right\}$$

Note that at $r = R$, $E_\theta(r=R) = 0$. The electric field must be normally incident onto a perfect conductor.

9. a) $\bar{f} = q[\bar{E}(\bar{r} + \bar{d}) - \bar{E}(\bar{r})]$

$$\bar{E}(\bar{r} + \bar{d}) \approx \bar{E}(\bar{r}) + \frac{\partial \bar{E}}{\partial x}\bigg|_{\bar{r}} d_x + \frac{\partial \bar{E}}{\partial y}\bigg|_{\bar{r}} d_y + \frac{\partial \bar{E}}{\partial z}\bigg|_{\bar{r}} d_z$$

$$\approx \bar{E}(\bar{r}) + (\bar{d}\cdot\nabla)\bar{E}$$

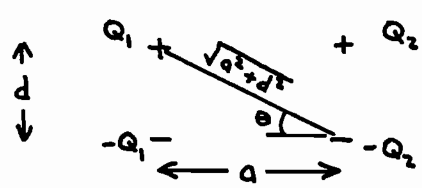
$$\bar{f} = q(\bar{d}\cdot\nabla)\bar{E} = (\bar{p}\cdot\nabla)\bar{E}$$

b) Colinear

$$E_{z2} = \frac{-p_2}{2\pi\varepsilon_o z^3}$$

$$\bar{f}_1 = p_1 \frac{\partial}{\partial z} E_{z2}\bar{i}_z = \frac{3p_1p_2}{2\pi\varepsilon_o z^4}\bigg|_{z=a} \bar{i}_z = \frac{3p_1p_2}{2\pi\varepsilon_o a^4}\,\bar{i}_z$$

Adjacent



$$f_{1r} = \frac{2Q_1Q_2}{4\pi\varepsilon_o}\left[\frac{-1}{a^2} + \frac{\cos\theta}{[a^2 + d^2]}\right]; \quad \cos\theta = \frac{a}{[a^2 + d^2]^{1/2}}$$

$$= \frac{Q_1Q_2}{2\pi\varepsilon_o}\left[-\frac{1}{a^2} + \frac{a}{[a^2 + d^2]^{3/2}}\right]$$

$$= \frac{Q_1Q_2}{2\pi\varepsilon_o}\left[-\frac{1}{a^2} + \frac{1}{a^2[1 + \frac{d^2}{a^2}]^{3/2}}\right]$$

$$\approx \frac{Q_1Q_2(-\frac{3}{2}d^2)}{2\pi\varepsilon_o a^4} = \frac{-3p_1p_2}{4\pi\varepsilon_o a^4}$$

c) Colinear

$$\bar{f}_1 = \frac{3p_1^2}{2\pi\varepsilon_o a^4}\bar{i}_z \sum_{n=1}^{\infty}\frac{1}{n^4}$$

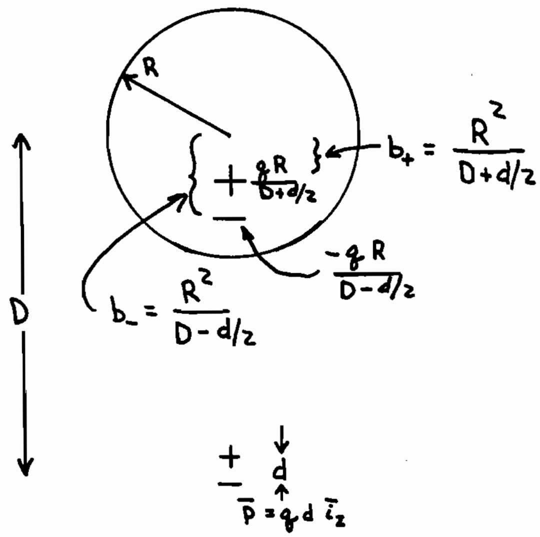
$$= \frac{\pi^3 p_1^2}{60\varepsilon_o a^4}\bar{i}_z$$

Adjacent

$$\bar{f}_1 = \frac{-3p^2}{4\pi\varepsilon_o a^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$= \frac{-p^2\pi^3}{120\varepsilon_o a^4}\bar{i}_r$$

10. a)



For d << D

$$b_- - b_+ \approx \frac{R^2}{D}\left[1 + \frac{d}{2D} - 1 + \frac{d}{2D}\right] \approx \frac{R^2 d}{D^2}$$

while each induced charge is approximately $\pm\frac{qR}{D}$. The resulting induced dipole moment is then

$$\bar{p}_{ind} = \frac{qR}{D}\frac{R^2 d}{D^2}\bar{i}_z = qd\frac{R^3}{D^3}\bar{i}_z = \bar{p}\frac{R^3}{D^3}.$$

b) The electric field along the z axis is just due to the superposition of fields due to both dipoles

$$E_z = \frac{p}{2\pi\varepsilon_o|z + D|^3} + \frac{p\,R^3/D^3}{2\pi\varepsilon_o|z + \frac{R^2}{D}|^3}$$

c) $\overline{f}_{sphere} = (\overline{p}_{ind}\cdot\nabla)\overline{E}$

$$= \frac{pR^3}{D^3}\frac{\partial}{\partial z}\left(\frac{p}{2\pi\varepsilon_o|z + D|^3}\right)\Bigg|_{z=-\frac{R^2}{D}}$$

$$= \frac{-3p^2R^3}{D^3 2\pi\varepsilon_o(z + D)^4}\Bigg|_{z=-\frac{R^2}{D}}$$

$$= \frac{-3p^2R^3D}{2\pi\varepsilon_o(D^2 - R^2)^4}$$

Section 3.2

11. $\nabla^2 V - \frac{V^2}{\ell_d^2} = 0;\quad \ell_d^2 = \frac{\varepsilon kT}{2\rho_o q}$

a) $$\frac{d^2V}{dx^2} - \frac{V^2}{\ell_d^2} = 0 \rightarrow V = \begin{cases} V_o e^{-x/\ell_d} & x > 0 \\ V_o e^{+x/\ell_d} & x < 0 \end{cases}$$

$$E_x = -\frac{dV}{dx} = \begin{cases} \frac{V_o}{\ell_d} e^{-x/\ell_d} & x > 0 \\ \frac{-V_o}{\ell_d} e^{+x/\ell_d} & x < 0 \end{cases}$$

$$\sigma_f = \varepsilon[E_x(x=0_+) - E_x(x=0_-)] = \frac{2\varepsilon V_o}{\ell_d} \rightarrow V_o = \frac{\sigma_f \ell_d}{2\varepsilon}$$

$$\rho_f = \frac{\varepsilon dE_x}{dx} = \frac{-\sigma_f}{2\ell_d} e^{-|x|/\ell_d}$$

b) $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) - \frac{V}{\ell_d^2} = 0 \rightarrow V = V_o K_o(r/\ell_d)$$

$$E_r = -\frac{dV}{dr} = \frac{V_o}{\ell_d} K_1(r/\ell_d)$$

$$\lim_{r\to 0} E_r = \frac{V_o}{r} = \frac{\lambda}{2\pi\varepsilon r} \to V_o = \frac{\lambda}{2\pi\varepsilon}$$

$$V = \frac{\lambda}{2\pi\varepsilon} K_o(r/\ell_d), \quad E_r = \frac{\lambda}{2\pi\varepsilon\ell_d} K_1(r/\ell_d)$$

$$\rho_f = \frac{\varepsilon}{r}\frac{\partial}{\partial r}(rE_r) = -\frac{\varepsilon V}{\ell_d^2} = \frac{-\lambda}{2\pi\ell_d^2} K_o(r/\ell_d)$$

c) $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) - \frac{V}{\ell_d^2} = 0 \to V = \frac{A_1}{r} e^{-r/\ell_d}$$

$$E_r = -\frac{dV}{dr} = \frac{A_1 e^{-r/\ell_d}}{r}\left(\frac{1}{r} + \frac{1}{\ell_d}\right)$$

$$E_r(r{=}R) = \frac{Q}{4\pi\varepsilon R^2} = \frac{A_1 e^{-R/\ell_d}}{R}\left(\frac{1}{R} + \frac{1}{\ell_d}\right) \to A_1 = \frac{Qe^{R/\ell_d}}{4\pi\varepsilon R\left(\frac{1}{R} + \frac{1}{\ell_d}\right)}$$

$$V = \frac{Qe^{-(r-R)/\ell_d}}{4\pi\varepsilon Rr\left(\frac{1}{R} + \frac{1}{\ell_d}\right)}, \quad E_r = \frac{Qe^{-(r-R)/\ell_d}\left(\frac{1}{r} + \frac{1}{\ell_d}\right)}{4\pi\varepsilon Rr\left(\frac{1}{R} + \frac{1}{\ell_d}\right)}$$

$$\rho_f = \frac{\varepsilon}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{-Qe^{-(r-R)/\ell_d}}{4\pi\ell_d r(R + \ell_d)}$$

12. a) $$\frac{d^2V}{dx^2} - \frac{V}{\ell_d^2} = 0 \to V = V_1 e^{x/\ell_d} + V_2 e^{-x/\ell_d}$$

$$V(x=-\ell) = -\frac{V_o}{2} = V_1 e^{-\ell/\ell_d} + V_2 e^{\ell/\ell_d}$$

$$V(x=\ell) = \frac{V_o}{2} = V_1 e^{\ell/\ell_d} + V_2 e^{-\ell/\ell_d}$$

$$V_1 = -V_2 = \frac{V_o}{4\sinh \ell/\ell_d}$$

$$V(x) = \frac{V_o}{4\sinh \ell/\ell_d}\left[e^{x/\ell_d} - e^{-x/\ell_d}\right]$$

$$= \frac{V_o}{2} \frac{\sinh x/\ell_d}{\sinh \ell/\ell_d}$$

$$E_x = -\frac{dV}{dx} = \frac{-V_o}{2\ell_d} \frac{\cosh x/\ell_d}{\sinh \ell/\ell_d}$$

$$\rho_f = \varepsilon \frac{dE_x}{dx} = \frac{-\varepsilon V_o}{2\ell_d^2} \frac{\sinh x/\ell_d}{\sinh \ell/\ell_d}$$

Positive charge near negative electrode and negative charge near positive electrode.

b) $\sigma_f(x=-\ell) = \varepsilon E_x(x=-\ell) = -\frac{\varepsilon V_o}{2\ell_d} \coth \ell/\ell_d$

$$\sigma_f(x=+\ell) = -\varepsilon E_x(x=\ell) = \frac{\varepsilon V_o}{2\ell_d} \coth \ell/\ell_d$$

$$\sigma_T = \int_{-\ell}^{+\ell} \rho_f dx = \frac{-\varepsilon V_o}{2\ell_d^2 \sinh \ell/\ell_d} \int_{-\ell}^{+\ell} \sinh x/\ell_d \, dx = 0$$

13. $m \frac{dv}{dt} = -m\nu v + qE_o$

a) $\frac{dv}{dt} + \nu v = \frac{q}{m} E_o$

$$v = A_1 e^{-\nu t} + \frac{q}{m\nu} E_o$$

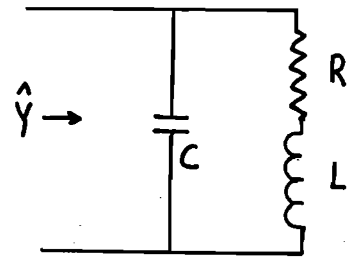
$$v(t=0) = 0 \rightarrow A_1 = -\frac{qE_o}{m\nu} \rightarrow v = \frac{qE_o}{m\nu}[1 - e^{-\nu t}]$$

b) $v = \frac{qE_o}{m\nu} e^{-\nu t}$

c) $v(t) = \mathrm{Re}\hat{V}e^{j\omega t} \rightarrow (j\omega + \nu)\hat{V} = \frac{q\hat{E}}{m}$; $\hat{E} = \frac{V_o}{s}$

$$\hat{I} = (qn\hat{V} + j\omega\hat{E})A = \left(\frac{q^2 n}{m(j\omega + \nu)} + j\omega\varepsilon\right) \frac{AV_o}{s}$$

$$\hat{Y} = \frac{\hat{I}}{V_o} = \frac{\varepsilon A}{s}\left(j\omega + \frac{\omega_p^2}{j\omega + \nu}\right); \quad \omega_p^2 = \frac{q^2 n}{m\varepsilon}$$



$$\hat{Y} = Cj\omega + \frac{1}{R + Lj\omega},$$

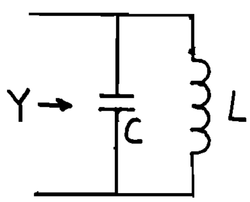
$$C = \frac{\varepsilon A}{s}, \quad L = \frac{s}{\varepsilon A\omega_p^2}, \quad R = \frac{\nu s}{\varepsilon A\omega_p^2}$$

14. $$J_c + \varepsilon \frac{\partial E}{\partial t} = \frac{i(t)}{A} \rightarrow \frac{\partial J_c}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2} = \frac{1}{A}\frac{di}{dt}$$

$$\frac{d^2E}{dt^2} + \omega_{pe}^2 E = \frac{1}{\varepsilon A}\frac{di}{dt}; \quad i(t) = \mathrm{Re}\,\hat{I}e^{j\omega t}; \quad E = \mathrm{Re}\,\hat{E}e^{j\omega t}, \quad \hat{E} = \frac{\hat{v}}{s}$$

$$\frac{\hat{v}}{s}[-\omega^2 + \omega_{pe}^2] = \frac{j\omega\hat{I}}{\varepsilon A} \rightarrow \hat{I} = \frac{\varepsilon A\hat{v}}{s}\frac{(\omega_{pe}^2 - \omega^2)}{j\omega}$$

$$Y = \frac{\hat{I}}{\hat{v}} = \frac{\varepsilon A}{s}\left(\frac{\omega_{pe}^2 - \omega^2}{j\omega}\right) = \frac{1 - \dfrac{\omega^2}{\omega_{pe}^2}}{\dfrac{j\omega s}{\varepsilon A\omega_{pe}^2}}$$



$$Y = Cj\omega + \frac{1}{Lj\omega} = \frac{1 - LC\omega^2}{Lj\omega} = \frac{\hat{I}}{\hat{v}}$$

$$L = \frac{s}{\varepsilon A\omega_{pe}^2}, \quad LC = \frac{1}{\omega_{pe}^2} \rightarrow C = \frac{1}{L\omega_{pe}^2} = \frac{\varepsilon A}{s}$$

15. a) $\frac{i}{A} = \sigma E$

b) $$\Delta(mv) = \int qE\,dt = q\int \frac{i}{A\sigma}\,dt = \frac{q}{A\sigma}\underbrace{\int i\,dt}_{Q} = \frac{qQ}{A\sigma} = m\omega R$$

$$Q = \frac{m\omega RA\sigma}{q}$$

c) $$\frac{di}{dt} = \omega_p^2 \varepsilon EA$$

$$\Delta(mv) = \frac{q}{\omega_p^2\varepsilon A}\int \frac{di}{dt} = \frac{qI}{\omega_p^2\varepsilon A} = m\omega R \rightarrow I = \frac{m\omega R\omega_p^2\varepsilon A}{q}$$

Section 3.3

16. a) For lossless dielectrics

$$\bar{n}\cdot(\bar{D}_2 - \bar{D}_1) = 0 \rightarrow \varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$$

$$\bar{n} \times (\bar{E}_2 - \bar{E}_1) = 0 \rightarrow E_1 \sin\theta_1 = E_2 \sin\theta_2$$

$$\varepsilon_1 \cot\theta_1 = \varepsilon_2 \cot\theta_2 \rightarrow \frac{\varepsilon_2}{\varepsilon_1} = \frac{\tan\theta_2}{\tan\theta_1}$$

$$E_2 = \frac{\varepsilon_1 E_1 \cos\theta_1}{\varepsilon_2 \cos\theta_2} = \frac{\varepsilon_1 E_1}{\varepsilon_2} \cos\theta_1 \sqrt{1 + \tan^2\theta_2}$$

$$= \frac{\varepsilon_1 E_1}{\varepsilon_2} \cos\theta_1 \left[1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \tan^2\theta_1\right]^{1/2}$$

b) For conductors

$$\bar{n}\cdot(\bar{J}_2 - \bar{J}_1) = 0 \rightarrow \sigma_1 E_1 \cos\theta_1 = \sigma_2 E_2 \cos\theta_2$$

so that we can replace the permittivities in (a) by the respective conductivities

$$\tan\theta_2 = \frac{\sigma_2}{\sigma_1}\tan\theta_1 \quad \Rightarrow \quad E_2 = \frac{\sigma_1 E_1 \cos\theta_1}{\sigma_2} \left[1 + \left(\frac{\sigma_2}{\sigma_1}\right)^2 \tan^2\theta_1\right]^{1/2}$$

$$\sigma_f = D_{2n} - D_{1n} = \varepsilon_2 E_2 \cos\theta_2 - \varepsilon_1 E_1 \cos\theta_1 = E_1 \cos\theta_1 [-\varepsilon_1 + \varepsilon_2 \frac{\sigma_1}{\sigma_2}]$$

c) $$\bar{n}\cdot(\bar{J}_2 - \bar{J}_1) + \frac{\partial \sigma_f}{\partial t} = 0 \quad \Rightarrow \quad \bar{n}\cdot[\bar{J}_2 - \bar{J}_1 + \frac{\partial}{\partial t}(\bar{D}_2 - \bar{D}_1)] = 0$$

$$(\sigma_2 + j\omega\varepsilon_2)\,\hat{E}_2 \cos\theta_2 = (\sigma_1 + j\omega\varepsilon_1)\,\hat{E}_1 \cos\theta_1$$

$$\hat{E}_2 \sin\theta_2 = \hat{E}_1 \sin\theta_1$$

$$\tan\theta_2 = \frac{(\sigma_2 + j\omega\varepsilon_2)\tan\theta_1}{(\sigma_1 + j\omega\varepsilon_1)}$$

$$\hat{\sigma}_f = \varepsilon_2 \hat{E}_2 \cos\theta_2 - \varepsilon_1 \hat{E}_1 \cos\theta_1 = \hat{E}_1 \cos\theta_1 \left(-\varepsilon_1 + \frac{\varepsilon_2(\sigma_1 + j\omega\varepsilon_1)}{(\sigma_2 + j\omega\varepsilon_2)}\right)$$

17. a) $$D_r = \frac{\lambda}{2\pi r} \quad 0 < r < \infty; \qquad E_r = \begin{cases} \dfrac{\lambda}{2\pi\varepsilon r} & 0 < r < a \\ \dfrac{\lambda}{2\pi\varepsilon_o r} & a < r < \infty \end{cases}$$

$$P_r = (\varepsilon - \varepsilon_o)E_r = \begin{cases} \dfrac{(\varepsilon - \varepsilon_o)\lambda}{2\pi\varepsilon r} & 0 < r < a \\ 0 & a < r < \infty \end{cases}$$

$$\sigma_p(r=a) = -[P_r(r=a_+) - P_r(r=a_-)] = \frac{(\varepsilon - \varepsilon_o)\lambda}{2\pi\varepsilon a}$$

$$\lambda_p(r=0) = -\frac{(\varepsilon - \varepsilon_o)\lambda}{\varepsilon}$$

b) $$D_y = \pm\frac{\sigma_f}{2} \quad \begin{matrix} y > 0 \\ y < 0 \end{matrix}; \qquad E_y = \begin{cases} \pm\dfrac{\sigma_f}{2\varepsilon} & \begin{matrix} 0 < y < \frac{d}{2} \\ -\frac{d}{2} < y < 0 \end{matrix} \\[2ex] \pm\dfrac{\sigma_f}{2\varepsilon_o} & \begin{matrix} y > \frac{d}{2} \\ y < -\frac{d}{2} \end{matrix} \end{cases}$$

$$P_y = (\varepsilon - \varepsilon_o)E_y = \begin{cases} \pm\dfrac{(\varepsilon - \varepsilon_o)\sigma_f}{2\varepsilon} & \begin{matrix} 0 < y < \frac{d}{2} \\ -\frac{d}{2} < y < 0 \end{matrix} \\[2ex] 0 & |y| > \frac{d}{2} \end{cases}$$

$$\sigma_p(y = \pm\tfrac{d}{2}) = -[P_y(y = \tfrac{d_+}{2}) - P_y(y = \tfrac{d_-}{2})] = \frac{(\varepsilon - \varepsilon_o)\sigma_f}{2\varepsilon}$$

$$\sigma_p(y=0) = -\frac{(\varepsilon - \varepsilon_o)\sigma_f}{\varepsilon}$$

c) $$D_r = \begin{cases} \dfrac{Qr}{4\pi R^3} \\[2ex] \dfrac{Q}{4\pi r^2} \end{cases}; \qquad E_r = \frac{D_r}{\varepsilon} = \begin{cases} \dfrac{Qr}{4\pi\varepsilon R^3} & r < R \\[2ex] \dfrac{Q}{4\pi\varepsilon_o r^2} & r > R \end{cases}$$

$$P_r = (\varepsilon - \varepsilon_o)E_r = \begin{cases} \dfrac{(\varepsilon - \varepsilon_o)Qr}{4\pi\varepsilon R^3} & r < R \\[2ex] 0 & r > R \end{cases}$$

$$\sigma_p(r=R) = -[P_r(r=R_+) - P_r(r=R_-)] = \frac{(\varepsilon - \varepsilon_o)Q}{4\pi\varepsilon R^2}$$

$$\rho_p(r) = -\nabla\cdot\overline{P} = \frac{-1}{r^2}\frac{\partial}{\partial r}(r^2P_r) = -\frac{(\varepsilon - \varepsilon_o)Q}{\frac{4}{3}\pi R^3\,\varepsilon}$$

18. $\overline{P} = P_o\overline{i}_z = P_o(\overline{i}_r\cos\theta - \overline{i}_\theta\,\sin\theta)$

a) $\sigma_p = -P_r(r{=}R) = -P_o\cos\theta$

b) $d\lambda_p = \sigma_p R d\theta = -P_o R\cos\theta d\theta,\ dq = -2\pi P_o R^2\sin\theta\cos\theta d\theta$

c) $dE_z = -\dfrac{dq\cos\theta}{4\pi\varepsilon_o R^2} = \dfrac{P_o}{2\varepsilon_o}\cos^2\theta\sin\theta d\theta$

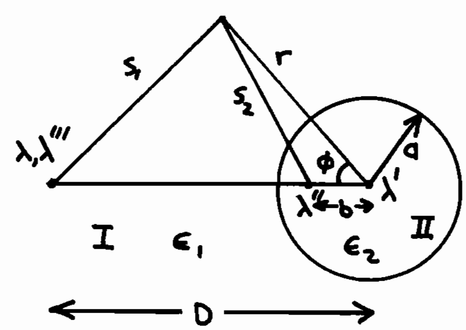
d) $E_z = \dfrac{P_o}{2\varepsilon_o}\displaystyle\int_0^\pi \cos^2\theta\sin\theta d\theta + E_o$

Let $u = \cos\theta,\ du = -\sin\theta d\theta$

$$E_z = -\frac{P_o}{2\varepsilon_o}\int_1^{-1} u^2 du + E_o$$

$$= -\frac{P_o}{6\varepsilon_o}u^3\Big|_1^{-1} + E_o = \frac{P_o}{3\varepsilon_o} + E_o$$

19. a) $V_I = \dfrac{-1}{2\pi\varepsilon_1}[\lambda\ln s_1 + \lambda'\ln r + \lambda''\ln s_2]$



$V_{II} = \dfrac{-1}{2\pi\varepsilon_2}\lambda'''\ \ln s_1$

where

$$s_1 = [D^2 + r^2 - 2rD\cos\phi]^{1/2}$$

$$s_2 = [b^2 + r^2 - 2rb\cos\phi]^{1/2};\quad b = \frac{a^2}{D}$$

$$\overline{E}_I = -\nabla V_I = \frac{1}{2\pi\varepsilon_1}\Big\{\frac{\lambda}{s_1^2}[(r - D\cos\phi)\overline{i}_r + D\sin\phi\overline{i}_\phi]$$

$$+ \frac{\lambda''}{s_2^2}[(r - b\cos\phi)\overline{i}_r + b\sin\phi\overline{i}_\phi] + \frac{\lambda'}{r}\overline{i}_r\Big\}$$

$$\overline{E}_{II} = -\nabla V_{II} = \frac{\lambda'''}{2\pi\varepsilon_2 s_1^2}\left[(r - D\cos\phi)\overline{i}_r + D\sin\phi\overline{i}_\phi\right]$$

At the cylinder surface the boundary conditions are

$$E_{\phi I}(r{=}a) = E_{\phi II}(r{=}a) \rightarrow \frac{1}{\varepsilon_1}\left(\frac{\lambda D}{s_1^2} + \frac{\lambda'' b}{s_2^2}\right) = \frac{\lambda''' D}{\varepsilon_2 s_1^2}$$

$$\varepsilon_1 E_{rI}(r{=}a) = \varepsilon_2 E_{rII}(r{=}a) \rightarrow \frac{\lambda'}{a} + \frac{\lambda}{s_1^2}(a - D\cos\phi) + \frac{\lambda''}{s_2^2}(a - b\cos\phi) = \frac{\lambda'''}{s_1^2}(a - D\cos\phi)$$

With $b = \frac{a^2}{D}$ at $r = a$, $s_2 = \frac{a}{D} s_1$ so that the boundary conditions reduce to

$$\left.\begin{aligned} &\frac{(\lambda + \lambda'')}{\varepsilon_1} = \frac{\lambda'''}{\varepsilon_2} \\ &2\lambda' + \lambda + \lambda'' = \lambda''' \end{aligned}\right\} \longrightarrow \begin{aligned} &\lambda' = -\lambda'' = \frac{\lambda(\varepsilon_2 - \varepsilon_1)}{\varepsilon_1 + \varepsilon_2} \\ &\lambda''' = \frac{2\varepsilon_2\lambda}{\varepsilon_1 + \varepsilon_2} \end{aligned}$$

$$\lambda'(D^2 + a^2) + \lambda a^2 + \lambda'' D^2 = \lambda''' a^2$$

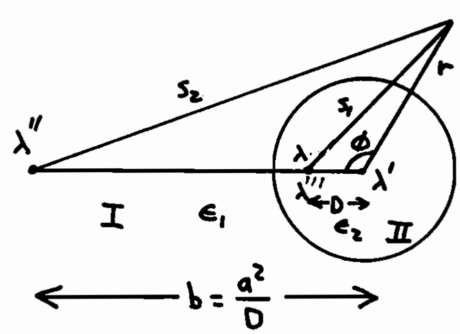
b) $$\lim_{\substack{a\to\infty \\ D=d+a}} b = \frac{a^2}{D} \approx a - d$$

Then λ' is at infinity having no effect while λ'' is now a distance d back from the surface as for a flat plane. The image charges λ'' and λ''' depend on the permittivities in the same way as in (3.3.3).

c) $$F_x = \frac{-\lambda}{2\pi\varepsilon_1}\left[\frac{\lambda'}{D} + \frac{\lambda''}{D - b}\right]$$

$$= \frac{\lambda^2}{2\pi\varepsilon_1}\frac{(\varepsilon_2 - \varepsilon_1)}{(\varepsilon_1 + \varepsilon_2)}\frac{a^2}{D(D^2 - a^2)}$$

d) $$V_I = \frac{-1}{2\pi\varepsilon_1}\left[\lambda' \ln r + \lambda''' \ln s_1\right]$$



$$V_{II} = \frac{-1}{2\pi\varepsilon_2}\left[\lambda \ln s_1 + \lambda'' \ln s_2\right]$$

$$\overline{E}_I = -\nabla V_I = \frac{1}{2\pi\varepsilon_1}\left\{\frac{\lambda'}{r}\overline{i}_r + \frac{\lambda'''}{s_1^2}\left[(r - D\cos\phi)\overline{i}_r + D\sin\phi\overline{i}_\phi\right]\right\}$$

$$\overline{E}_{II} = -\nabla V_{II} = \frac{1}{2\pi\varepsilon_2}\left\{\frac{\lambda}{s_1^2}\left[(r - D\cos\phi)\overline{i}_r + D\sin\phi\overline{i}_\phi\right] + \frac{\lambda''}{s_2^2}\left[(r - b\cos\phi)\overline{i}_r + b\sin\phi\overline{i}_\phi\right]\right\}$$

$$\left.\begin{array}{l} E_{\phi I}(r=a) = E_{\phi II}(r=a) \\ \\ \varepsilon_1 E_{rI}(r=a) = \varepsilon_2 E_{rII}(r=a) \end{array}\right\} \rightarrow \begin{array}{l} \lambda' = \lambda'' = \dfrac{\lambda(\varepsilon_2 - \varepsilon_1)}{\varepsilon_1 + \varepsilon_2} \\ \\ \lambda''' = \dfrac{2\varepsilon_1\lambda}{\varepsilon_1 + \varepsilon_2} \end{array}$$

$$F_x = \frac{\lambda\lambda''}{2\pi\varepsilon_2\left(D - \frac{a^2}{D}\right)}$$

$$= \frac{\lambda^2(\varepsilon_2 - \varepsilon_1)\,D}{2\pi\varepsilon_2(\varepsilon_1 + \varepsilon_2)(D^2 - a^2)}$$

20. a) $\sigma_1 E_y(0_+) = \sigma_2 E_y(0_-)$

$E_x(0_+) = E_x(0_-)$

b) The solution is the same as in (3.3.3) if we replace the permittivity by the respective conductivities

$$q' = -\frac{(\sigma_2 - \sigma_1)q}{\sigma_1 + \sigma_2}, \quad q'' = \frac{2\sigma_2\, q}{\sigma_1 + \sigma_2}$$

c) $$f_y = \frac{qq'}{4\pi\varepsilon_1(2d)^2}$$

$$= \frac{-q^2(\sigma_2 - \sigma_1)}{16\pi\varepsilon_1 d^2(\sigma_1 + \sigma_2)}$$

21. a) When the electrodes are shorted $E = 0$, $D = P_o = \sigma_f \rightarrow q = P_o A$.

b) When the upper electrode is moved to infinity, $D \rightarrow 0$ so that all the charge flows off the electrode onto the capacitor.

$$v_c = \frac{q}{C} = \frac{P_o A}{C}$$

22. a) $\sigma_p(z = \pm \frac{L}{2}) = \frac{P_o}{2}$

$$\rho_P(z) = -\nabla\cdot\overline{P} = -\frac{\partial P_z}{\partial z} = -\frac{P_o}{L}$$

b) Superposing and spatially shifting the results of (2.3.5b) letting $\sigma_o = \frac{P_o}{2}$ and of (2.3.5d) letting $\rho_o = -P_o/L$ yields

$$D_z = \frac{P_o}{2}\left\{\frac{-2(z-\frac{L}{2})}{\left[a^2+(z-\frac{L}{2})^2\right]^{1/2}} - \frac{2(z+\frac{L}{2})}{\left[a^2+(z+\frac{L}{2})^2\right]^{1/2}} - \frac{\left[a^2+(z-\frac{L}{2})^2\right]^{1/2}}{L} + \frac{\left[a^2+(z+\frac{L}{2})^2\right]^{1/2}}{L}\right\}$$

$$E_z = \begin{cases} \dfrac{D_z}{\varepsilon_o} & |z| > \dfrac{L}{2} \\[2ex] \dfrac{D_z - P_o z/L}{\varepsilon_o} & |z| < \dfrac{L}{2} \end{cases}$$

23. a) $\oint \overline{D}\cdot d\overline{S} = D_r\, 4\pi r^2 = 0 \Rightarrow D_r = 0$

$$D_r = \begin{cases} \varepsilon_o E_r + P_r & r < R \\ \varepsilon_o E_r & r > R \end{cases}$$

$$E_r = \begin{cases} \dfrac{-P_r}{\varepsilon_o} = \dfrac{-P_o r}{\varepsilon_o R} & r < R \\[2ex] 0 & r > R \end{cases}$$

b) $E_o(s-b) + E_P b = V_o;\quad \varepsilon_o E_o = \varepsilon_o E_P + P_o$

$$E_o = \frac{V_o}{s} + \frac{P_o b}{\varepsilon_o s},\quad E_P = \frac{V_o}{s} - \frac{P_o(s-b)}{\varepsilon_o s}$$

24. $E_b b + E_a a = v(t) \rightarrow E_a = \dfrac{v(t)}{a} - \dfrac{E_b b}{a}$

$$J_a = \sigma E_a, \quad \frac{\partial J_b}{\partial t} = \omega_{pe}^2 \, \varepsilon_b E_b$$

$$J_a - J_b + \frac{\partial}{\partial t}[\varepsilon_a E_a - \varepsilon_b E_b] = 0$$

$$\frac{\partial^2 E_b}{\partial t^2} + \alpha \frac{\partial E_b}{\partial t} + \omega_o^2 E_b = \frac{\varepsilon_a}{\varepsilon_a b + \varepsilon_b a} \frac{\partial^2 v}{\partial t^2} + \frac{\sigma}{\varepsilon_a b + \varepsilon_b a} \frac{\partial v}{\partial t}$$

$$\alpha = \frac{\sigma \, b}{\varepsilon_a b + \varepsilon_b a}, \quad \omega_o^2 = \frac{\omega_{pe}^2 \, \varepsilon_b a}{\varepsilon_a b + \varepsilon_b a}$$

a) $E_b = \hat{E} e^{st}$

$$s^2 + \alpha s + \omega_o^2 = 0 \rightarrow s = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \omega_o^2}$$

Critically damped when $\omega_o^2 = \frac{\alpha^2}{4}$

$$E_b = [\hat{E}_1 \sin\beta t + \hat{E}_2 \cos\beta t] e^{-\alpha t/2}; \quad \beta = \sqrt{\omega_o^2 - \frac{\alpha^2}{4}}$$

$$\frac{\partial E_b}{\partial t} = \{\beta[\hat{E}_1 \cos\beta t - \hat{E}_2 \sin\beta t] - \frac{\alpha}{2}[\hat{E}_1 \sin\beta t + \hat{E}_2 \cos\beta t]\} e^{-\alpha t/2}$$

Initial conditions:

$$\left.\begin{aligned} E_b(t=0) &= \frac{\varepsilon_a V_o}{\varepsilon_a b + \varepsilon_b a} = \hat{E}_2 \\ \frac{\partial E_b}{\partial t}\Big|_{t=0} &= -\alpha E_b(t=0) + \frac{\sigma V_o}{\varepsilon_a b + \varepsilon_b a} = \beta \hat{E}_1 - \frac{\alpha \hat{E}_2}{2} \end{aligned}\right\} \rightarrow \hat{E}_1 = \frac{V_o[\sigma - \frac{\alpha \varepsilon_a}{2}]}{\beta(\varepsilon_a b + \varepsilon_b a)}$$

$$\sigma_f = \varepsilon_a E_a - \varepsilon_b E_b$$

$$i(t) = \ell D[\sigma E_a + \varepsilon_a \frac{\partial E_a}{\partial t}] = \ell D[J_b + \varepsilon_b \frac{\partial E_b}{\partial t}]$$

b) $v(t) = \mathrm{Re}\, V_o e^{j\omega t}, \quad E_b = \mathrm{Re}\, \hat{E}_b e^{j\omega t}$

$$\hat{E}_b = \frac{\dfrac{\varepsilon_a V_o}{\varepsilon_a b + \varepsilon_b a}\left[-\omega^2 + \dfrac{\sigma}{\varepsilon_a} j\omega\right]}{-\omega^2 + \alpha j\omega + \omega_o^2}, \qquad \hat{E}_a = \frac{V_o}{a} - \frac{\hat{E}_b b}{a}$$

$$\hat{\sigma}_f = \varepsilon_a \hat{E}_a - \varepsilon_b \hat{E}_b, \qquad \hat{i} = \ell D[\sigma + j\omega\varepsilon_a]\hat{E}_a = \ell D\left[\frac{\omega_{pe}^2 \varepsilon_b}{j\omega} + j\omega\varepsilon_b\right]\hat{E}_b$$

Section 3.4

25. a) Series

$$J = \sigma_1 E_1 = \sigma_2 E_2 = \frac{i}{\ell D}$$

$$E_1 b + E_2 a = V_o$$

$$\frac{i}{\ell D}\left[\frac{b}{\sigma_1} + \frac{a}{\sigma_2}\right] = V_o$$

$$R = \frac{V_o}{i} = \left[\frac{b}{\sigma_1} + \frac{a}{\sigma_2}\right]\frac{1}{\ell d}$$

Parallel

$$i = \sigma_1 E_1 aD + \sigma_2 E_2 bD$$

$$E_1 = E_2 = \frac{V_o}{s}$$

$$i = \frac{DV_o}{s}[\sigma_1 a + \sigma_2 b]$$

$$R = \frac{V_o}{i} = \frac{s}{D[\sigma_1 a + \sigma_2 b]}$$

b) Series

$$E_1 = \frac{A_1}{r}, \quad E_2 = \frac{A_2}{r}$$

$$A_1 \ln\frac{R_2}{R_1} + A_2 \ln\frac{R_3}{R_2} = V_o$$

$$\sigma_1 A_1 = \sigma_2 A_2$$

$$A_2 = \frac{\sigma_1}{\sigma_2} A_1 = \frac{\sigma_1 V_o}{\left[\sigma_1 \ln\dfrac{R_3}{R_2} + \sigma_2 \ln\dfrac{R_2}{R_1}\right]}$$

$$i = J_r 2\pi r\ell = \sigma_2 2\pi\ell A_2$$

Parallel

$$E_r = \frac{V_o}{r \ln\dfrac{R_2}{R_1}}$$

$$i = [\sigma_1(2\pi - \alpha) + \sigma_2\alpha]E_r r\ell$$

$$= \frac{V_o \ell}{\ln\dfrac{R_2}{R_1}}[\sigma_1(2\pi - \alpha) + \sigma_2\alpha]$$

$$R = \frac{V_o}{i} = \frac{\ln\dfrac{R_2}{R_1}}{\ell[\sigma_1(2\pi - \alpha) + \sigma_2\alpha]}$$

$$= \frac{\sigma_1\sigma_2 2\pi\ell\, V_o}{[\sigma_1 \ell n \frac{R_3}{R_2} + \sigma_2 \ell n \frac{R_2}{R_1}]}$$

$$R = \frac{V_o}{i} = \frac{\sigma_1 \ell n \frac{R_3}{R_2} + \sigma_2 \ell n \frac{R_2}{R_1}}{2\pi\ell\sigma_1\sigma_2}$$

c) $E_1 = \frac{A_1}{r^2}$, $E_2 = \frac{A_2}{r^2}$

$$-A_1(\frac{1}{R_2} - \frac{1}{R_1}) - A_2(\frac{1}{R_3} - \frac{1}{R_2}) = V_o$$

$$\sigma_1 A_1 = \sigma_2 A_2$$

$$A_2 = \frac{\sigma_1 A_1}{\sigma_2} = \frac{-V_o\sigma_1}{\sigma_1(\frac{1}{R_3} - \frac{1}{R_2}) + \sigma_2(\frac{1}{R_2} - \frac{1}{R_1})}$$

$$i = J_r 4\pi r^2$$

$$= \sigma_2 E_2 4\pi r^2$$

$$= \sigma_2 A_2 4\pi$$

$$= \frac{4\pi\sigma_2\sigma_1 V_o}{\sigma_1(\frac{1}{R_2} - \frac{1}{R_3}) + \sigma_2(\frac{1}{R_1} - \frac{1}{R_2})}$$

$$R = \frac{V_o}{i}$$

$$= \frac{\sigma_1(\frac{1}{R_2} - \frac{1}{R_3}) + \sigma_2(\frac{1}{R_1} - \frac{1}{R_2})}{4\pi\sigma_1\sigma_2}$$

$$E_r = \frac{V_o}{(\frac{1}{R_1} - \frac{1}{R_2})r^2}$$

$$i = 2\pi E_r r^2[\sigma_2[1-\cos\frac{\alpha}{2}] + \sigma_1[\cos\frac{\alpha}{2} + 1]]$$

$$= \frac{2\pi V_o}{(\frac{1}{R_1} - \frac{1}{R_2})}[\sigma_2(1-\cos\frac{\alpha}{2}) + \sigma_1(\cos\frac{\alpha}{2} + 1)]$$

$$R = \frac{V_o}{i}$$

$$= \frac{(\frac{1}{R_1} - \frac{1}{R_2})}{2\pi[\sigma_2(1 - \cos\frac{\alpha}{2}) + \sigma_1(1 + \cos\frac{\alpha}{2})]}$$

26. a) $\sigma(x)E(x) = J_o \rightarrow E(x) = \dfrac{J_o}{\sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}}$

$$\int_0^s E(x)dx = \frac{J_o s}{\sigma_2 - \sigma_1} \ell n[\sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}]\Big|_0^s = \frac{J_o s}{\sigma_2 - \sigma_1} \ell n \frac{\sigma_2}{\sigma_1} = V_o$$

$$R = \frac{V_o}{i} = \frac{V_o}{J_o \ell D} = \frac{s}{\ell D(\sigma_2 - \sigma_1)} \ell n \frac{\sigma_2}{\sigma_1}$$

b) $$E(x) = \frac{V_o(\sigma_2 - \sigma_1)}{s \ell n \frac{\sigma_2}{\sigma_1}[\sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}]}$$

$$\rho_f = \varepsilon \frac{dE}{dx} = - \frac{\varepsilon V_o(\sigma_2 - \sigma_1)^2}{s^2 \ell n \frac{\sigma_2}{\sigma_1}[\sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}]^2}$$

$$\sigma_f(x=0) = \varepsilon E(x=0) = \frac{\varepsilon V_o(\sigma_2 - \sigma_1)}{s\sigma_1 \ell n \frac{\sigma_2}{\sigma_1}}$$

$$\sigma_f(x=s) = -\varepsilon E(x=s) = \frac{-\varepsilon V_o(\sigma_2 - \sigma_1)}{s\sigma_2 \ell n \frac{\sigma_2}{\sigma_1}}$$

c) $$Q = \ell D \int_0^s \rho_f dx = \frac{-\varepsilon V_o(\ell D)(\sigma_2 - \sigma_1)^2}{s^2 \ell n \frac{\sigma_2}{\sigma_1}} \int_0^s \frac{dx}{[\sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}]^2}$$

$$= \frac{\varepsilon V_o \ell D(\sigma_2 - \sigma_1)}{s \ell n \frac{\sigma_2}{\sigma_1}[\sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}]}\Bigg|_0^s$$

$$= \frac{\varepsilon V_o \ell D(\sigma_2 - \sigma_1)}{s \ell n \frac{\sigma_2}{\sigma_1}} \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right)$$

$$= -\ell D[\sigma_f(x=0) + \sigma_f(x=s)]$$

27. Rectangular

$$R_1 = \frac{[a+b]}{\sigma A},\quad R_2 = \frac{[a^2+b^2]^{1/2}}{\sigma A} \rightarrow R = \frac{\frac{R_1}{2} R_2}{\frac{R_1}{2} + R_2} = \frac{R_1 R_2}{R_1 + 2R_2}$$

Circular

$$R_1 = \frac{\pi R}{\sigma A},\quad R_2 = \frac{2R}{\sigma A} \rightarrow R = \frac{\frac{R_1}{2} R_2}{\frac{R_1}{2} + R_2} = \frac{R_1 R_2}{R_1 + 2R_2} = \frac{2\pi R}{\sigma A[\pi + 4]}$$

Section 3.5

28. $RC = \varepsilon/\sigma \rightarrow R = \dfrac{\varepsilon}{\sigma C}$

$$C = \frac{2\pi\varepsilon\ell}{\cosh^{-1}\left[\pm\dfrac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right]} \longrightarrow R = \frac{\cosh^{-1}\left[\pm\dfrac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right]}{2\pi\sigma\ell}$$

29. Series

a) $D = \varepsilon_1 E_1 = \varepsilon_2 E_2 = \sigma_f(x=0) = \dfrac{q}{\ell d}$

$E_1 b + E_2 a = V_o \rightarrow \dfrac{q}{\ell d}\left[\dfrac{b}{\varepsilon_1} + \dfrac{a}{\varepsilon_2}\right] = V_o$

$C = \dfrac{q}{V_o} = \dfrac{\ell d}{\dfrac{b}{\varepsilon_1} + \dfrac{a}{\varepsilon_2}}$

b) $E_1 = \dfrac{A_1}{r},\quad E_2 = \dfrac{A_2}{r}$

$A_1 \ln\dfrac{R_2}{R_1} + A_2 \ln\dfrac{R_3}{R_2} = V_o$

$\varepsilon_1 A_1 = \varepsilon_2 A_2 \rightarrow A_2 = \dfrac{\varepsilon_1}{\varepsilon_2} A_1 = \dfrac{\varepsilon_1 V_o}{\varepsilon_1 \ln\dfrac{R_3}{R_2} + \varepsilon_2 \ln\dfrac{R_2}{R_1}}$

Parallel

$E = \dfrac{V_o}{s}$

$q = \dfrac{V_o d}{s}[\varepsilon_1 a + \varepsilon_2 b]$

$C = \dfrac{q}{V_o} = \dfrac{d}{s}(\varepsilon_1 a + \varepsilon_2 b)$

$E = \dfrac{V_o}{r \ln\dfrac{b}{a}}$

$q(R_1) = E(R_1)\ell R_1[\varepsilon_2\alpha + \varepsilon_1(2\pi - \alpha)]$

$= \dfrac{V_o \ell}{\ln\dfrac{R_2}{R_1}}[\varepsilon_2\alpha + \varepsilon_1(2\pi - \alpha)]$

$$q(R_1) = \varepsilon_1 E_1(R_1) 2\pi R_1 \ell$$

$$= \varepsilon_1 A_1 2\pi\ell$$

$$C = \frac{q(R_1)}{V_o} = \frac{\varepsilon_1\varepsilon_2 2\pi\ell}{\varepsilon_1 \ln\frac{R_3}{R_2} + \varepsilon_2 \ln\frac{R_2}{R_1}}$$

c) $E_1 = \frac{A_1}{r^2}, \quad E_2 = \frac{A_2}{r^2}$

$$A_1\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + A_2\left(\frac{1}{R_2} - \frac{1}{R_3}\right) = V_o$$

$$\varepsilon_1 A_1 = \varepsilon_2 A_2 = \frac{\varepsilon_1\varepsilon_2 V_o}{\varepsilon_1\left(\frac{1}{R_2} - \frac{1}{R_3}\right) + \varepsilon_2\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

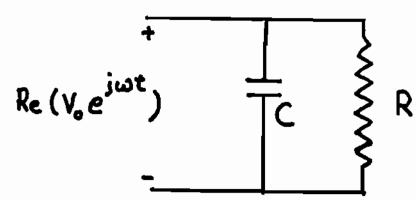
$$C = \frac{q(R_1)}{V_o} = \frac{4\pi\varepsilon_1\varepsilon_2}{\varepsilon_1\left(\frac{1}{R_2} - \frac{1}{R_3}\right) + \varepsilon_2\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

$$C = \frac{q(R_1)}{V_o} = \frac{[\varepsilon_2\alpha + \varepsilon_1(2\pi - \alpha)]\ell}{\ln\frac{R_2}{R_1}}$$

$$E_r = \frac{V_o}{r^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

$$C = \frac{q(R_1)}{V_o} = \frac{2\pi}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}\left[\varepsilon_2\left(1 - \cos\frac{\alpha}{2}\right) + \varepsilon_1\left(1 + \cos\frac{\alpha}{2}\right)\right]$$

30. $R = \frac{V}{I}, \quad C = \frac{\varepsilon}{\sigma R}$



$$Y(j\omega) = Cj\omega + \frac{1}{R} = \frac{RCj\omega + 1}{R}$$

$$\hat{I} = V_o Y(j\omega) = \frac{V_o}{R}[RCj\omega + 1]$$

$$i(t) = \mathrm{Re}\,\hat{I}e^{j\omega t} = \frac{V_o}{R}\cos\omega t - V_o C\omega\sin\omega t$$

31. $\nabla\cdot\overline{D} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r) = 0 \rightarrow D_r = \frac{A}{r}$

$$E_r = \frac{D_r}{\varepsilon(r)} = \frac{A}{r\left[\varepsilon_1 + (\varepsilon_2 - \varepsilon_1)\frac{(r-a)}{b-a}\right]}$$

$$\int_a^b E_r dr = V_o = \frac{-A}{\left[\varepsilon_1 - \frac{(\varepsilon_2-\varepsilon_1)a}{b-a}\right]} \ln \left(\frac{\varepsilon_1 + (\varepsilon_2-\varepsilon_1)\frac{(r-a)}{(b-a)}}{r}\right)\Bigg|_{r=a}^{b}$$

$$= \frac{-A(b-a)}{\varepsilon_1 b - \varepsilon_2 a} \ln \frac{\varepsilon_2 a}{\varepsilon_1 b}$$

$$q(r=a) = D_r(r=a)2\pi a\ell = 2\pi A\ell = -\frac{2\pi\ell(\varepsilon_1 b - \varepsilon_2 a)V_o}{(b-a)\ln\frac{\varepsilon_2 a}{\varepsilon_1 b}}$$

$$C = \frac{q(r=a)}{V_o} = \frac{2\pi\ell(\varepsilon_2 a - \varepsilon_1 b)}{(b-a)\ln\frac{\varepsilon_2 a}{\varepsilon_1 b}}$$

Section 3.6

32. a) $\rho_f(t) = \rho_o e^{-t/\tau}$, $E(x) = \begin{cases} \dfrac{\rho_f(t)x}{\varepsilon_o} + C_1 & 0 \le x \le b \\ C_2 & b \le x \le a+b \end{cases}$; $\tau = \varepsilon_o/\sigma$

$$\int_0^{a+b} E(x)dx = \frac{\rho_f b^2}{2\varepsilon_o} + C_1 b + C_2 a = V_o$$

$$\bar{n}\cdot[\bar{J}_a - \bar{J}_b] + \frac{\partial\sigma_f}{\partial t} = 0 \rightarrow -\sigma\left[\frac{\rho_f(t)b}{\varepsilon_o} + C_1\right] + \frac{\partial}{\partial t}\left[\varepsilon_o\left[C_2 - C_1 - \frac{\rho_f(t)b}{\varepsilon_o}\right]\right] = 0$$

$$\frac{\partial C_2}{\partial t} + \frac{C_2 a}{(a+b)\tau} = \frac{V_o}{(a+b)\tau} \rightarrow C_2(t) = Ae^{-\frac{a}{a+b}\frac{t}{\tau}} + \frac{V_o}{a}$$

$$\sigma_f(t=0) = \varepsilon_o\left(C_2 - \frac{\rho_f(t=0)b}{\varepsilon_o} - C_1\right) = 0 \rightarrow A + \frac{V_o}{a} - \frac{\rho_o b}{\varepsilon_o} - C_1 = 0$$

$$A = \frac{\rho_o b^2}{2\varepsilon_o(a+b)} - \frac{V_o b}{a(a+b)}$$

$$C_2(t) = \left[\frac{\rho_o b^2}{2\varepsilon_o(a+b)} - \frac{V_o b}{a(a+b)}\right] e^{-\frac{a}{a+b}\frac{t}{\tau}} + \frac{V_o}{a}$$

$$C_1(t) = \frac{V_o}{b} - \frac{\rho_f(t)b}{2\varepsilon_o} - \frac{C_2 a}{b} = \frac{-\rho_o b e^{-t/\tau}}{2\varepsilon_o} - \left[\frac{\rho_o ab}{2\varepsilon_o(a+b)} - \frac{V_o}{a+b}\right] e^{-\frac{a}{a+b}\frac{t}{\tau}}$$

b) $\sigma_f(x=b) = \varepsilon_o[E(x=b_+) - E(x=b_-)]$

$$= \varepsilon_o[C_2 - \frac{\rho_f(t)b}{\varepsilon_o} - C_1]$$

$$= \varepsilon_o \left[\frac{V_o}{a} + (\frac{\rho_o b}{2\varepsilon_o} - \frac{V_o}{a})e^{-\frac{a}{a+b}\frac{t}{\tau}} - \frac{\rho_o b}{2\varepsilon_o} e^{-t/\tau}\right]$$

c) $f_x = \int_0^{b_-} \rho_f(x)E dx + \frac{1}{2}\sigma_f(x=b)[E(x=b_+) + E(x=b_-)]$

$$= \int_{0_+}^{b_-} \varepsilon_o E \frac{\partial E}{\partial x} dx + \frac{\varepsilon_o}{2}[E(x=b_+) - E(x=b_-)][E(x=b_+) + E(x=b_-)]$$

$$= \frac{1}{2}\varepsilon_o[E^2(x=b_-) - E^2(x=0_+)] + \frac{\varepsilon_o}{2}[E^2(x=b_+) - E^2(x=b_-)]$$

$$= \frac{1}{2}\varepsilon_o[E^2(x=b_+) - E^2(x=0_+)]$$

$$= \frac{1}{2}\varepsilon_o[C_2^2 - C_1^2]$$

33.

$$E_r = \begin{cases} \dfrac{\rho_o r^2}{3\varepsilon a_o} e^{-t/\tau} & 0 < r < a_o \\ \dfrac{\rho_o a_o^2}{3\varepsilon r} e^{-t/\tau} & a_o < r < a_1 \; ; \quad \tau = \varepsilon/\sigma \\ \dfrac{\rho_o a_o^2}{3\varepsilon_o r} & r > a_1 \end{cases}$$

$$\sigma_f = \varepsilon_o E_r(r=a_{1+}) - \varepsilon E_r(r=a_{1-})$$

$$= \frac{\rho_o a_o^2}{3a_1}(1 - e^{-t/\tau})$$

34. a) $$E_r = \begin{cases} \dfrac{C_1}{r} & R_1 < r < R_2 \\[2ex] \dfrac{C_2}{r} & R_2 < r < R_3 \end{cases} \rightarrow \int_{R_1}^{R_3} E_r dr = C_1 \ln\frac{R_2}{R_1} + C_2 \ln\frac{R_3}{R_2} = V_o$$

At $t = 0$, $\sigma_f(r=R_2) = 0$

$$\varepsilon_1 C_1 = \varepsilon_2 C_2 = \frac{\varepsilon_1 \varepsilon_2 V_o}{\varepsilon_1 \ln\dfrac{R_3}{R_2} + \varepsilon_2 \ln\dfrac{R_2}{R_1}}$$

b) $$J_r(r=R_{2+}) = J_r(r=R_{2-}) \rightarrow \sigma_1 C_1 = \sigma_2 C_2 = \frac{\sigma_1 \sigma_2 V_o}{\sigma_1 \ln\dfrac{R_3}{R_2} + \sigma_2 \ln\dfrac{R_2}{R_1}}$$

c) $$J_r(r=R_{2+}) - J_r(r=R_{2-}) + \frac{\partial \sigma_f}{\partial t} = 0$$

$$\sigma_2 C_2 - \sigma_1 C_1 + \frac{\partial}{\partial t}[\varepsilon_2 C_2 - \varepsilon_1 C_1] = 0$$

$$\frac{\partial C_1}{\partial t} + \frac{C_1}{\tau} = \frac{\sigma_2 V_o}{\varepsilon_1 \ln\dfrac{R_3}{R_2} + \varepsilon_2 \ln\dfrac{R_2}{R_1}} \; ; \quad \tau = \frac{\varepsilon_1 \ln\dfrac{R_3}{R_2} + \varepsilon_2 \ln\dfrac{R_2}{R_1}}{\sigma_1 \ln\dfrac{R_3}{R_2} + \sigma_2 \ln\dfrac{R_2}{R_1}}$$

$$C_1 = \frac{\sigma_2 V_o[1 - e^{-t/\tau}]}{\sigma_1 \ln\dfrac{R_3}{R_2} + \sigma_2 \ln\dfrac{R_2}{R_1}} + \frac{\varepsilon_2 V_o e^{-t/\tau}}{\varepsilon_1 \ln\dfrac{R_3}{R_2} + \varepsilon_2 \ln\dfrac{R_2}{R_1}}$$

$$C_2 = \frac{V_o - C_1 \ln\dfrac{R_2}{R_1}}{\ln\dfrac{R_3}{R_2}} = \frac{\sigma_1 V_o}{\sigma_1 \ln\dfrac{R_3}{R_2} + \sigma_2 \ln\dfrac{R_2}{R_1}} + \frac{V_o \ln\dfrac{R_2}{R_1}}{\ln\dfrac{R_3}{R_2}} \left(\frac{\varepsilon_2}{\varepsilon_1 \ln\dfrac{R_3}{R_2} + \varepsilon_2 \ln\dfrac{R_2}{R_1}} - \frac{\sigma_2}{\sigma_1 \ln\dfrac{R_3}{R_2} + \sigma_2 \ln\dfrac{R_2}{R_1}} \right) e^{-t/\tau}$$

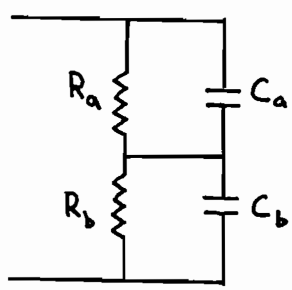
$$\sigma_f(r=b) = \frac{\varepsilon_2 C_2 - \varepsilon_1 C_1}{b}$$

d)

$$E_r = \begin{cases} \mathrm{Re}\, \dfrac{\hat{C}_1 e^{j\omega t}}{r} & R_1 < r < R_2 \\ \mathrm{Re}\, \dfrac{\hat{C}_2 e^{j\omega t}}{r} & R_2 < r < R_3 \end{cases} \longrightarrow \hat{C}_1 \ln \frac{R_2}{R_1} + \hat{C}_2 \ln \frac{R_3}{R_2} = V_o$$

$$(\sigma_2 + j\omega\varepsilon_2)\hat{C}_2 = (\sigma_1 + j\omega\varepsilon_1)\hat{C}_1 = \frac{(\sigma_1 + j\omega\varepsilon_1)(\sigma_2 + j\omega\varepsilon_2)V_o}{(j\omega\varepsilon_1 + \sigma_1)\ln \frac{R_3}{R_2} + (j\omega\varepsilon_2 + \sigma_2)\ln \frac{R_2}{R_1}}$$

e)



$$R_a = \frac{\ln \frac{R_2}{R_1}}{2\pi\sigma_1 \ell}, \quad R_b = \frac{\ln \frac{R_3}{R_2}}{2\pi\sigma_2 \ell}$$

$$C_a = \frac{2\pi\varepsilon_1 \ell}{\ln \frac{R_2}{R_1}}, \quad C_b = \frac{2\pi\varepsilon_2 \ell}{\ln \frac{R_3}{R_2}}$$

35. $\nabla\cdot(\rho_f \overline{U}) + \frac{\sigma}{\varepsilon}\rho_f = 0 \rightarrow \rho_f \nabla\cdot\overline{U} + (\overline{U}\cdot\nabla)\rho_f + \frac{\sigma}{\varepsilon}\rho_f = 0$

$$\overline{U} = \frac{A}{r^2}\overline{i}_r \rightarrow \nabla\cdot\overline{U} = 0 \rightarrow \frac{A}{r^2}\frac{d\rho_f}{dr} + \frac{\sigma}{\varepsilon}\rho_f = 0$$

$$\frac{d\rho_f}{\rho_f} = \frac{-\sigma}{\varepsilon A} r^2 dr \rightarrow \ln\rho_f = \frac{-\sigma r^3}{3\varepsilon A} + C$$

$$\rho_f = \rho_o e^{-\sigma r^3/3\varepsilon A}$$

$$\nabla\cdot\overline{E} = \frac{\rho_f}{\varepsilon} \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{\rho_o}{\varepsilon} e^{-\sigma r^3/3\varepsilon A}$$

$$E_r = \frac{-\rho_o A}{\sigma r^2} e^{-\sigma r^3/3\varepsilon A} + \frac{C_1}{r^2}$$

$$E_r(r=0)\text{ finite} \rightarrow E_r = \frac{\rho_o A}{\sigma r^2}(1 - e^{-\sigma r^3/3\varepsilon A})$$

36. a) $\rho_f(x) = \rho_o e^{-x/\ell_m} = \varepsilon \frac{dE}{dx}$; $\ell_m = \frac{\varepsilon U}{\sigma}$

$$E = \frac{-\rho_o \ell_m}{\varepsilon} e^{-x/\ell_m} + C$$

$$\int_0^\ell E dx = C\ell + \frac{\rho_o \ell_m^2}{\varepsilon}(e^{-\ell/\ell_m} - 1) = V_o$$

$$E = \frac{-\rho_o \ell_m}{\varepsilon} e^{-x/\ell_m} + \frac{V_o}{\ell} - \frac{\rho_o \ell_m^2}{\varepsilon \ell}(e^{-\ell/\ell_m} - 1)$$

b) $$f_x = \int_0^\ell \rho_f E A dx$$

$$= \int_0^\ell A\varepsilon E \frac{dE}{dx} dx$$

$$= \frac{1}{2}\varepsilon A[E^2(\ell) - E^2(0)]$$

c) $$\frac{i}{A} = \sigma E + \rho_f U \rightarrow E = \frac{i}{A\sigma} - \frac{\rho_o U}{\sigma} e^{-x/\ell_m}$$

$$\int_0^\ell E dx = \frac{i\ell}{A\sigma} + \frac{\rho_o U \ell_m}{\sigma}(e^{-\ell/\ell_m} - 1) = -iR_L$$

$$i = \frac{\rho_o U \ell_m}{\sigma} \frac{(1 - e^{-\ell/\ell_m})}{R_L + \frac{\ell}{A\sigma}}$$

37. $v(z,t=0_-) = V_o \dfrac{\cosh\sqrt{2RG}\,(z - \ell)}{\cosh\sqrt{2RG}\,\ell}$; $i(z,t=0_-) = V_o \sqrt{\dfrac{G}{2R}} \dfrac{\sinh\sqrt{2RG}\,(z - \ell)}{\cosh\sqrt{2RG}\,\ell}$

If the $z = 0$ end is either open or short circuited, the steady state voltage and current distributions are zero. The transient voltage and current are of the form

$$v(z,t) = \hat{V}(z)e^{-\alpha t}, \quad i(z,t) = \hat{I}(z)e^{-\alpha t}$$

with

$$\hat{V}(z) = V_1 \sin pz + V_2 \cos pz$$

$$\hat{I}(z) = -\frac{1}{2R}\frac{d\hat{V}}{dz} = \frac{-p}{2R}[V_1 \cos pz - V_2 \sin pz]; \quad p^2 = 2RC\alpha - 2RG$$

Because the line is open circuited at $z = \ell$,

$$\hat{i}(z=\ell) = 0 \rightarrow V_1 \cos p\ell - V_2 \sin p\ell = 0$$

a) Open circuit at $z = 0$

$$\hat{I}(z=0) = 0 = V_1 \rightarrow \sin p\ell = 0 \rightarrow p\ell = n\pi$$

$$i(z,t) = \sum_{n=1}^{\infty} I_n \sin \frac{n\pi z}{\ell} e^{-\alpha_n t}; \quad \alpha_n = \frac{(\frac{n\pi}{\ell})^2 + 2RG}{2RC}$$

Initial conditions:

$$i(z,t=0) = V_o \sqrt{\frac{G}{2R}} \frac{\sinh\sqrt{2RG}\,(z-\ell)}{\cosh\sqrt{2RG}\,\ell} = \sum_{n=1}^{\infty} I_n \sin \frac{n\pi z}{\ell}$$

$$I_n = -\frac{2V_o \sqrt{\frac{G}{2R}}\, n\pi \tanh\sqrt{2RG}\,\ell}{\ell^2[(\frac{n\pi}{\ell})^2 + 2RG]}$$

$$v(z,t) = \sum_{n=1}^{\infty} \frac{2RI_n}{(n\pi/\ell)} \cos \frac{n\pi z}{\ell} e^{-\alpha_n t}$$

b) Short circuit at $z = 0$

$$\hat{V}(z=0) = 0 = V_2 \rightarrow \hat{V}(z) = V_1 \sin pz$$

$$\hat{I}(z) = -\frac{1}{2R}\frac{d\hat{V}}{dz} = -\frac{pV_1}{2R}\cos pz$$

$$\hat{I}(z=\ell) = 0 \rightarrow p\ell = (2n+1)\frac{\pi}{2}, \quad n = 0, 1, 2, \ldots$$

Then

$$v(z,t) = \sum_{n=0}^{\infty} V_n \sin \frac{(2n+1)\pi z}{2\ell} e^{-\alpha_n t}$$

$$v(z,t=0) = V_o \frac{\cosh\sqrt{2RG}\,(z-\ell)}{\cosh\sqrt{2RG}\,\ell} = \sum_{n=0}^{\infty} V_n \sin\frac{(2n+1)\pi z}{2\ell}$$

$$V_n = \frac{V_o\pi(2n+1)}{\ell^2\left[2RG + \left(\frac{(2n+1)\pi}{2\ell}\right)^2\right]}$$

$$i(z,t) = -\frac{1}{2R}\frac{\partial v}{\partial z} = \frac{-\pi}{4R\ell}\sum_{n=0}^{\infty}(2n+1)V_n\cos(2n+1)\frac{\pi z}{2\ell}\,e^{-\alpha_n t}$$

38. a) $v(z) = V_1\sinh\sqrt{2RG}\,z + V_2\cosh\sqrt{2RG}\,z$

$$v(z=0) = V_o \rightarrow v(z) = -\frac{V_o\sinh\sqrt{2RG}\,(z-\ell)}{\sinh\sqrt{2RG}\,\ell}$$

$$v(z=\ell) = 0$$

$$i(z) = -\frac{1}{2R}\frac{dv}{dz} = V_o\sqrt{\frac{G}{2R}}\,\frac{\cosh\sqrt{2RG}\,(z-\ell)}{\sinh\sqrt{2RG}\,\ell}$$

b) $$v(z,t) = -\frac{V_o\sinh\sqrt{2RG}\,(z-\ell)}{\sinh\sqrt{2RG}\,\ell} + \hat{v}(z)e^{-\alpha t}$$

$$\hat{v}(z) = V_1\sin pz + V_2\cos pz, \quad p^2 = 2RC\alpha - 2RG$$

$$\left.\begin{aligned}\hat{v}(z=0) &= 0\\ \hat{v}(z=\ell) &= 0\end{aligned}\right\} \rightarrow \hat{v}(z) = \sum_{n=1}^{\infty} V_n\sin\frac{n\pi z}{\ell}, \quad p = \frac{n\pi}{\ell}$$

$$v(z,t) = -\frac{V_o\sinh\sqrt{2RG}\,(z-\ell)}{\sinh\sqrt{2RG}\,\ell} + \sum_{n=1}^{\infty} V_n\sin\frac{n\pi z}{\ell}\,e^{-\alpha_n t}$$

$$v(z,t=0) = 0 \rightarrow V_n = \frac{-2n\pi V_o}{\ell^2\left[2RG + \left(\frac{n\pi}{\ell}\right)^2\right]}$$

$$i(z,t) = -\frac{1}{2R}\frac{\partial v}{\partial z} = V_o\sqrt{\frac{G}{2R}}\,\frac{\cosh\sqrt{2RG}\,(z-\ell)}{\sinh\sqrt{2RG}\,\ell} - \frac{1}{2R}\sum_{n=1}^{\infty}\frac{n\pi}{\ell}V_n\cos\frac{n\pi z}{\ell}\,e^{-\alpha_n t}$$

39. $-\frac{\partial v}{\partial z} = 2iR$

$$-\frac{\partial i}{\partial z} = C\frac{\partial v}{\partial t} + Gv \quad \rightarrow \quad \frac{\partial^2 v}{\partial z^2} = 2RC\frac{\partial v}{\partial t} + 2RGv$$

$$v(z,t) = \text{Re}\,\hat{v}(z)e^{j\omega t}$$

$$\frac{d^2\hat{v}}{dz^2} - 2R[Cj\omega + G]\hat{v} = 0 \rightarrow \hat{v}(z) = V_1 \sin pz + V_2 \cos pz$$

$$p^2 = -2R(G + Cj\omega)$$

a) Short circuited at $z = \ell$

$$\left.\begin{aligned}\hat{v}(z=\ell) &= 0\\ \hat{v}(z=0) &= V_o\end{aligned}\right\} \rightarrow \hat{v}(z) = -\frac{V_o \sin p(z-\ell)}{\sin p\ell}$$

$$\hat{i}(z) = -\frac{1}{2R}\frac{d\hat{v}}{dz} = \frac{p}{2R}V_o\frac{\cos p(z-\ell)}{\sin p\ell}$$

Open circuited at $z = \ell$

$$\hat{i}(z) = -\frac{p}{2R}[V_1 \cos pz - V_2 \sin pz]$$

$$\left.\begin{aligned}\hat{v}(z=0) &= V_o\\ \hat{i}(z=\ell) &= 0\end{aligned}\right\} \rightarrow \hat{v}(z) = V_o\frac{\cos p(z-\ell)}{\cos p\ell}$$

$$\hat{i}(z) = -\frac{pV_o}{2R}\frac{\sin p(z-\ell)}{\cos p\ell}$$

b) Short Circuit

$$p\ell = n\pi \rightarrow j\omega = -\frac{(\frac{n\pi}{\ell})^2}{2RC} - \frac{G}{C}$$

Open Circuit

$$p\ell = (2n+1)\frac{\pi}{2} \rightarrow j\omega = -\frac{[\frac{(2n+1)\pi}{2\ell}]^2}{2RC} - \frac{G}{C}$$

c) $\langle P\rangle = \frac{1}{2}\text{Re}[\hat{v}(z=0)\hat{i}^*(z=0)]$

Short circuit:

$$\langle P\rangle = \frac{V_o^2}{4R} \operatorname{Re}[p^* \cot p^* \ell]$$

Open circuit:

$$\langle P\rangle = \frac{V_o^2}{4R} \operatorname{Re}[p^* \tan p^* \ell]$$

40. $R_{inner} = \frac{\ell}{\sigma_c \pi a^2}$, $R_{outer} = \frac{\ell}{\sigma_c \pi (c^2 - b^2)} \rightarrow 2R = R_{inner} + R_{outer} = \frac{\ell}{\sigma_c \pi}\left[\frac{1}{a^2} + \frac{1}{c^2 - b^2}\right]$

$$C = \frac{2\pi\varepsilon\ell}{\ln\frac{b}{a}}, \quad G = \frac{2\pi\sigma\ell}{\ln\frac{b}{a}}$$

Section 3.7

41. a) $\nabla\cdot\bar{J} + \frac{\partial\rho_f}{\partial t} = 0; \quad \bar{J} = \rho_f \mu\bar{E}$

$$\rho_f = \varepsilon\nabla\cdot\bar{E} \rightarrow \rho_f = \varepsilon\frac{\partial E}{\partial x}$$

$$\nabla\cdot[\rho_f \mu\bar{E} + \varepsilon\frac{\partial\bar{E}}{\partial t}] = 0 \rightarrow \frac{\partial}{\partial x}[\varepsilon\mu E\frac{\partial E}{\partial x} + \varepsilon\frac{\partial E}{\partial t}] = 0$$

$$\underbrace{\varepsilon\mu E\frac{\partial E}{\partial x}}_{\substack{\text{conduction}\\ \text{current}\\ \text{density}}} + \underbrace{\varepsilon\frac{\partial E}{\partial t}}_{\substack{\text{displacement}\\ \text{current}\\ \text{density}}} = \underbrace{J(t)}_{\substack{\text{total}\\ \text{current}\\ \text{density}}}$$

b) $\frac{\varepsilon\mu}{2}[E^2(\ell) - E^2(0)] + \varepsilon\frac{dv}{dt} = J(t)\ell$

c) Ahead of the front $\rho_f = 0 \rightarrow J(t) = \varepsilon\frac{dE(\ell)}{dt} = \frac{\varepsilon\mu}{2\ell}E^2(\ell)$

where $E(0) = 0$ (SCL) and $\frac{dv}{dt} = 0$ for $t > 0$

$$\frac{dE(\ell)}{E^2(\ell)} = \frac{\mu dt}{2\ell} \rightarrow -\frac{1}{E(\ell)} = \frac{\mu t}{2\ell} + C$$

$$E(\ell, t=0) = \frac{V_o}{\ell} \rightarrow E(\ell) = \frac{V_o/\ell}{1 - \dfrac{\mu t V_o}{2\ell^2}}$$

d) $E(x=s,t) = E(\ell,t)$

e) $$\frac{ds}{dt} = \mu E(s,t) = \frac{\mu V_o/\ell}{1 - \dfrac{\mu t V_o}{2\ell^2}} \rightarrow s = -2\ell\, \ell n[1 - \frac{\mu V_o t}{2\ell^2}]$$

f) $$s = \ell = -2\ell\, \ell n[1 - \frac{\mu V_o \tau}{2\ell^2}] \rightarrow \tau = \frac{2\ell^2}{\mu V_o}(1 - e^{-1/2})$$

g), h) <u>Steady state:</u>

$$\varepsilon\mu E \frac{\partial E}{\partial x} = J_o \rightarrow E = \sqrt{\frac{2J_o x}{\varepsilon\mu} + E_o^2}$$

$$\rho_f = \frac{J_o}{\mu E} = \frac{J_o}{\mu\sqrt{\dfrac{2J_o x}{\varepsilon\mu} + E_o^2}}$$

$$E = -\frac{dV}{dx} \rightarrow V = -\frac{\varepsilon\mu}{3J_o}\left\{[\frac{2J_o x}{\varepsilon\mu} + E_o^2]^{3/2} - E_o^3\right\}$$

where $V(x=0) = 0$

$$V(x=\ell) = -V_o = -\frac{\varepsilon\mu}{3J_o}\left\{[\frac{2J_o \ell}{\varepsilon\mu} + E_o^2]^{3/2} - E_o^3\right\}$$

$$J_o = \frac{\varepsilon\mu V_o^2}{16\ell^3}\left\{9 - 12\frac{E_o\ell}{V_o} + [(9 - 12\frac{E_o\ell}{V_o})^2 + 192(1 - \frac{E_o\ell}{V_o})(\frac{E_o\ell}{V_o})^3]^{1/2}\right\}$$

$$\text{SCL} \rightarrow E_o = 0 \rightarrow J_o = \frac{9}{8}\frac{\varepsilon\mu V_o^2}{\ell^3}$$

42. a) $\rho_f \mu E_r = \dfrac{I}{2\pi r\ell}$; $\nabla\cdot\overline{E} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(rE_r) = \dfrac{\rho_f}{\varepsilon}$

$$E_r \frac{\partial}{\partial r}(rE_r) = \frac{I}{2\pi\ell\varepsilon\mu} \rightarrow (rE_r)^2 = \frac{I}{2\pi\ell\varepsilon\mu}(r^2 - R_i^2) + R_i^2 E_i^2$$

$$E_r = \frac{1}{r}[\frac{I}{2\pi\ell\varepsilon\mu}(r^2 - R_i^2) + R_i^2 E_i^2]^{1/2}; \quad \rho_f = \frac{I}{2\pi r\ell\mu E_r}$$

b) $E_i = 0 \rightarrow E_r = \frac{1}{r}\left[\frac{I}{2\pi\varepsilon\mu\ell}\right]^{1/2}[r^2 - R_i^2]^{1/2}$

$$\int_{R_i}^{R_o} E_r dr = \left[\frac{I}{2\pi\varepsilon\mu\ell}\right]^{1/2}\left\{\sqrt{r^2 - R_i^2} - R_i\cos^{-1}\frac{R_i}{r}\right\}\Bigg|_{r=R_i}^{R_o} = V_o$$

$$I = \frac{2\pi\varepsilon\mu\ell V_o^2}{\left\{\sqrt{R_o^2 - R_i^2} - R_i\cos^{-1}\frac{R_i}{R_o}\right\}^2}$$

c) $E_i^2 = \frac{I}{2\pi\varepsilon\mu\ell} \rightarrow E(r) = E_i \rightarrow \int_{R_i}^{R_o} E(r)dr = E_i(R_o - R_i) = V_o$

$$E_i^2 = \left(\frac{V_o}{R_o - R_i}\right)^2 = \frac{I}{2\pi\varepsilon\mu\ell} \rightarrow I = 2\pi\varepsilon\mu\ell\left(\frac{V_o}{R_o - R_i}\right)^2$$

d) $\rho_f\mu E_r = \frac{I}{4\pi r^2}; \quad \nabla\cdot\bar{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) = \frac{\rho_f}{\varepsilon}$

$$E_r\frac{\partial}{\partial r}(r^2E_r) = \frac{I}{4\pi\varepsilon\mu} \rightarrow \frac{(r^2E_r)^2}{2} = \frac{I(r^3 - R_i^3)}{12\pi\varepsilon\mu} + \frac{(R_i^2E_i)^2}{2}$$

$$E_r = \left(\frac{I}{6\pi\varepsilon\mu}\right)^{1/2}\frac{1}{r^2}\left[r^3 - R_i^3 + \frac{R_i^4E_i^2}{I}6\pi\varepsilon\mu\right]^{1/2}$$

$$\rho_f = \frac{I}{4\pi r^2\mu E_r}$$

Section 3.8

43. a) $W = \frac{1}{2}q[V(\bar{r} + \bar{d}) - V(\bar{r})]$

$$\approx \frac{1}{2}q\nabla V\cdot\bar{d}$$

$$\approx -\frac{1}{2}\bar{p}\cdot\bar{E}$$

b) $W = -\frac{1}{2}p\frac{(2p)(2)}{4\pi\varepsilon_o s^3}\sum_{n=1}^{\infty}\frac{1}{n^3} \approx -\frac{1.2\,p^2}{2\pi\varepsilon_o s^3}$

c) $W = -\frac{1}{2}\frac{p(2p)(2)}{4\pi\varepsilon_o s^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} \approx \frac{.9\ p^2}{2\pi\varepsilon_o s^3}$

d) $W = -\frac{1}{2}\,p\,\left(\frac{-p}{4\pi\varepsilon_o s^3}\right)(2)\sum_{n=1}^{\infty}\frac{1}{n^3} \approx \frac{1.2\ p^2}{4\pi\varepsilon_o s^3}$ (identical)

$W = -\frac{1}{2}\,p\,\left(\frac{-p}{4\pi\varepsilon_o s^3}\right)(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} \approx \frac{-.9\ p^2}{4\pi\varepsilon_o s^3}$ (alternating polarity)

44. $\overline{E} = \frac{p}{4\pi\varepsilon_o r^3}[2\cos\theta\overline{i}_r + \sin\theta\overline{i}_\theta] \rightarrow \frac{1}{2}\varepsilon_o|\overline{E}|^2 = \frac{p^2}{32\pi^2\varepsilon_o r^6}[4\cos^2\theta + \sin^2\theta]$

$$W = \int_{\theta=0}^{\pi}\int_{r=R}^{\infty}\frac{1}{2}\varepsilon_o|\overline{E}|^2\, 2\pi r^2\sin\theta\, dr\, d\theta$$

$$= \frac{p^2}{16\pi\varepsilon_o}\int_{\theta=0}^{\pi}\int_{r=R}^{\infty}\frac{[4\cos^2\theta + \sin^2\theta]}{r^4}\sin\theta\, dr\, d\theta$$

$$= \frac{p^2}{48\pi\varepsilon_o R^3}\int_{\theta=0}^{\pi}[4\cos^2\theta + \sin^2\theta]\sin\theta\, d\theta$$

$$= \frac{p^2}{48\pi\varepsilon_o R^3}\left[-\frac{4\cos^3\theta}{3} - \frac{1}{3}\cos\theta(\sin^2\theta + 2)\right]\Bigg|_{\theta=0}^{\pi}$$

$$= \frac{p^2}{12\pi\varepsilon_o R^3}$$

45. a) Each drop carries a charge $\frac{Q}{N}$ with radius r so that the volume is conserved

$$N\frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3 \rightarrow r = \frac{R}{N^{1/3}}$$

b) $W_e = \frac{N(\frac{Q}{N})^2}{8\pi\varepsilon_o(R/N^{1/3})} = \frac{Q^2}{8\pi\varepsilon_o RN^{2/3}}$

c) $W_{final} = NW_S\, 4\pi\left(\frac{R}{N^{1/3}}\right)^2 + W_e = 4\pi R^2 W_S N^{1/3} + \frac{Q^2}{8\pi\varepsilon_o RN^{2/3}}$

d) $W_{original} = W_S 4\pi R^2 + \dfrac{Q^2}{8\pi\varepsilon_o R}$

$$\text{Work} = W_{final} - W_{original} = W_S 4\pi R^2[N^{1/3} - 1] + \frac{Q^2}{8\pi\varepsilon_o R}(N^{-2/3} - 1)$$

e) $\dfrac{\partial \text{Work}}{\partial N} = \dfrac{W_S 4\pi R^2}{3N^{2/3}} - \dfrac{2}{3}\dfrac{Q^2}{8\pi\varepsilon_o R N^{5/3}} = 0 \rightarrow N = \dfrac{Q^2}{16\pi^2\varepsilon_o^2 R^2 W_S}$

$$N = \frac{10^{-12}}{16\pi^2(8.854 \times 10^{-12})(10^{-6})(.072)} \approx 10^4 \text{ droplets of radius } r = \frac{10^{-3}}{10^{4/3}} \approx 4.64 \times 10^{-5} \text{ meters}$$

46. a) $E_r = \begin{cases} \dfrac{\sigma_a a}{\varepsilon r} & a < r < b \\ 0 & r > b \end{cases} \rightarrow W = \int_a^b \frac{1}{2}\varepsilon E_r^2 2\pi r dr$

$$= \frac{\pi(\sigma_a a)^2}{\varepsilon}\int_a^b \frac{dr}{r}$$

$$= \frac{\pi(\sigma_a a)^2}{\varepsilon}\ln\frac{b}{a}$$

b) $E_r = \begin{cases} \dfrac{\rho_a r}{2\varepsilon} & 0 < r < a \\ \dfrac{\rho_a a^2}{2\varepsilon r} & a < r < b \end{cases} \rightarrow W = \int_0^a \frac{1}{2}\varepsilon(\frac{\rho_a r}{2\varepsilon})^2 2\pi r dr + \int_a^b \frac{1}{2}\varepsilon(\frac{\rho_a a^2}{2\varepsilon r})2\pi r dr$

$$= \frac{\pi\rho_a^2 a^4}{4\varepsilon}[\frac{1}{4} + \ln\frac{b}{a}]$$

c) $E_r = \begin{cases} \dfrac{\rho_a r}{2\varepsilon} & 0 < r < a \\ \dfrac{\rho_a a^2}{2\varepsilon r} + \dfrac{\rho_b(r^2 - a^2)}{2\varepsilon r} & a < r < b \end{cases}$

$$W = \int_0^a \frac{1}{2}\varepsilon(\frac{\rho_a r}{2\varepsilon})^2 2\pi r dr + \int_a^b \frac{1}{2}\varepsilon[\frac{\rho_a a^2}{2\varepsilon r} + \rho_b\frac{(r^2 - a^2)}{2\varepsilon r}]^2 2\pi r dr$$

$$= \frac{\pi\rho_a^2 a^4}{16\varepsilon} + \frac{\pi a^4}{4\varepsilon}(\rho_a - \rho_b)^2\ln\frac{b}{a} + \frac{\pi\rho_b^2}{16\varepsilon}(b^4 - a^4) + \frac{\pi\rho_b}{2\varepsilon}(\rho_a - \rho_b)a^2(b - a)$$

47. a) $W = \frac{1}{2} QV_-(r=0) - \frac{1}{2}Q[V_+(R) + V_-(R)]$

$$V_-(r) = \begin{cases} \dfrac{-Q}{4\pi\varepsilon_o r} & r \geq R \\[2ex] \dfrac{-Q}{4\pi\varepsilon_o R} & r \leq R \end{cases} \quad ; \quad V_+(r) = \frac{Q}{4\pi\varepsilon_o r}$$

$$W = \frac{1}{2} Q[\frac{-Q}{4\pi\varepsilon_o R}] - \frac{1}{2} Q[0]$$

$$= \frac{-Q^2}{8\pi\varepsilon_o R}$$

b) $$V(r) = \begin{cases} \dfrac{3Q}{8\pi\varepsilon_o R_1^3}(R_1^2 - \dfrac{r^2}{3}) - \dfrac{Q}{4\pi\varepsilon_o R_2} & r < R_1 \\[2ex] \dfrac{Q}{4\pi\varepsilon_o r} - \dfrac{Q}{4\pi\varepsilon_o R_2} & R_1 < r < R_2 \end{cases}$$

$$W = \frac{1}{2}\int_{r=0}^{R_1} \rho V\, r^2 \sin\theta\, dr\, d\theta\, d\phi + \frac{1}{2}\int \sigma_f(R_2)\, \overset{0}{\cancel{V(R_2)}}\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{2}\int_{r=0}^{R_1} \frac{Q}{\frac{4}{3}\pi R_1^3}\left[\frac{3Q}{8\pi\varepsilon_o R_1^3}(R_1^2 - \frac{r^2}{3}) - \frac{Q}{4\pi\varepsilon_o R_2}\right] 4\pi r^2 dr$$

$$= \frac{Q^2}{8\pi\varepsilon_o R_1}\left[\frac{6}{5} - \frac{R_1}{R_2}\right]$$

Check:

$$E_r = \begin{cases} \dfrac{Qr}{4\pi\varepsilon_o R_1^3} & 0 < r < R_1 \\[2ex] \dfrac{Q}{4\pi\varepsilon_o r^2} & R_1 < r < R_2 \\[2ex] 0 & r > R_2 \end{cases}$$

$$W = \frac{1}{2}\varepsilon_o \int_0^{R_2} |E_r|^2\, 4\pi r^2 dr$$

$$= 2\pi\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o}\right)^2\left\{\int_0^{R_1}\frac{r^4}{R_1^6}\,dr + \int_{R_1}^{R_2}\frac{1}{r^2}\,dr\right\} = \frac{Q^2}{8\pi\varepsilon_o R_1}\left[\frac{6}{5} - \frac{R_1}{R_2}\right]$$

48. a) $W_{init} = \frac{1}{2}Cv_1^2(t{=}0) + \frac{1}{2}Cv_2^2(t{=}0) = \frac{1}{2}CV_o^2$

b) $\frac{v_1 - v_2}{R} = i = \frac{Cdv_2}{dt} = -\frac{Cdv_1}{dt}\ ; \qquad R = \frac{\ell}{\sigma A}$

$$v_2 + v_1 = V_o \rightarrow \frac{dv_1}{dt} + \frac{2v_1}{RC} = \frac{V_o}{RC}$$

$$v_1 = \frac{V_o}{2}\left[1 + e^{-t/\tau}\right];\quad v_2 = \frac{V_o}{2}\left[1 - e^{-t/\tau}\right];\quad \tau = \frac{RC}{2}$$

$$i = \frac{V_o}{R}e^{-t/\tau}$$

c) $\lim\limits_{t\to\infty} v_1 = v_2 = \frac{V_o}{2},\quad i = 0$

$$W_{final} = \frac{1}{2}Cv_1^2 + \frac{1}{2}Cv_2^2 = \frac{1}{4}CV_o^2 = \frac{1}{2}W_{init}$$

d) $W_{diss} = \int_0^\infty i^2R\,dt$

$$= \frac{V_o^2}{R}\int_0^\infty e^{-2t/\tau}dt$$

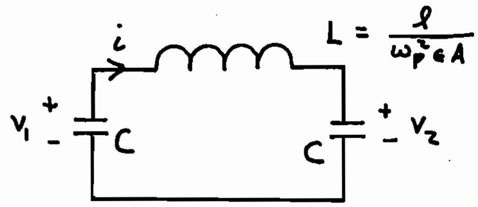
$$= -\frac{V_o^2\tau}{2R}e^{-2t/\tau}\Big|_0^\infty$$

$$= \frac{1}{4}CV_o^2 = \frac{1}{2}W_{init}$$

e) If $R = 0 \rightarrow \tau = 0$. From (d) the dissipated energy is independent of R. The circuit problem becomes nonphysical as an infinite current flows for zero time.

f) $\frac{di}{dt} = \omega_p^2\varepsilon AE \rightarrow \frac{Cd^2v_2}{dt^2} = -\frac{Cd^2v_1}{dt^2} = \frac{di}{dt} = \frac{\omega_p^2\varepsilon A}{\ell}(v_1 - v_2)$

$$v_1 + v_2 = V_o \rightarrow \frac{C d^2 v_1}{dt^2} + \frac{2\omega_p^2 \varepsilon A}{\ell} v_1 = \frac{\omega_p^2 \varepsilon A V_o}{\ell}$$



g) $$\frac{L d^2 i}{dt^2} + \frac{2i}{C} = 0 \rightarrow i(t) = I_1 \sin\omega_o t + I_2 \cos\omega_o t, \quad \omega_o = \sqrt{\frac{2}{LC}}$$

$$= \sqrt{\frac{2\omega_p^2 \varepsilon A}{\ell C}}$$

$$i(t=0) = 0 \rightarrow I_2 = 0$$

$$\rightarrow i(t) = \frac{V_o}{L\omega_o} \sin\omega_o t$$

$$L \left.\frac{di}{dt}\right|_{t=0} = V_o = L I_1 \omega_o$$

$$v_1(t) = -\frac{1}{C} \int i dt = \frac{V_o}{2} [\cos\omega_o t + 1]$$

$$v_2(t) = \frac{1}{C} \int i dt = -\frac{V_o}{2} [\cos\omega_o t - 1]$$

h) $$W_1 = \frac{1}{2} C v_1^2 = \frac{1}{8} C V_o^2 [1 + 2\cos\omega_o t + \cos^2\omega_o t]$$

$$W_2 = \frac{1}{2} C v_2^2 = \frac{1}{8} C V_o^2 [1 - 2\cos\omega_o t + \cos^2\omega_o t]$$

$$W_L = \frac{1}{2} L i^2 = \frac{1}{2} \frac{V_o^2}{L\omega_o^2} [\sin^2\omega_o t] = \frac{1}{4} C V_o^2 \sin^2\omega_o t$$

i) $$W_{total} = W_1 + W_2 + W_L = \frac{1}{4} C V_o^2 [1 + \sin^2\omega_o t + \cos^2\omega_o t]$$

$$= \frac{1}{2} C V_o^2$$

Section 3.9

49. a) $\overline{T} = \overline{r} \times \overline{F} \quad \rightarrow \quad T = qd\sin\theta E$

$$= d\overline{i_z} \times q\overline{E} \qquad = pE\sin\theta$$

b) $dW = Td\theta \quad \rightarrow \quad W = pE \int_0^\theta \sin\theta d\theta$

$$= pE\sin\theta d\theta \qquad = -pE(\cos\theta - 1)$$

c) $n = n_o e^{pE/kT} e^{pE\cos\theta/kT}$

$$N = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{R} nr^2 \sin\theta dr d\theta d\phi$$

$$= n_o e^{pE/kT} \frac{2\pi R^3}{3} \int_{\theta=0}^{\pi} e^{pE\cos\theta/kT} \sin\theta d\theta$$

Let $u = \frac{pE}{kT}\cos\theta$, $du = -\frac{pE}{kT}\sin\theta d\theta$

$$N = -n_o e^{pE/kT} \frac{2\pi R^3}{3} \frac{kT}{pE} \int_{pE/kT}^{-pE/kT} e^u du$$

$$= -\frac{2\pi R^3 kT}{3pE} n_o e^{pE/kT} [\underbrace{e^{-pE/kt} - e^{pE/kt}}_{-2\sinh\frac{pE}{kT}}]$$

$$n = \frac{3pEN}{4\pi R^3 kT\sinh\frac{pE}{kT}} e^{pE\cos\theta/kT}$$

d) $dP_z = pn\cos\theta d\theta r^2 \sin\theta dr d\phi$

e) $P_z = \frac{pN}{2\sinh\frac{pE}{kT}} \int_{\theta=0}^{\pi} \frac{pE}{kT} \cos\theta \sin\theta e^{pE\cos\theta/kT} d\theta$

$$= \frac{-pN}{2\sinh\frac{pE}{kT}} \int_{pE/kT}^{-pE/kT} \frac{kT}{pE} ue^u du$$

$$= \frac{-pN}{2\sinh\frac{pE}{kT}} \frac{kT}{pE} e^{u}(u - 1) \Bigg|_{u = pE/kT}^{-pE/kT}$$

$$= \frac{-pN}{2\sinh\frac{pE}{kT}} \frac{kT}{pE} [e^{-pE/kT}(-\frac{pE}{kT} - 1) - e^{pE/kT}(\frac{pE}{kT} - 1)]$$

$$= \frac{-pN}{\sinh\frac{pE}{kT}} \frac{kT}{pE} [-\frac{pE}{kT}\cosh\frac{pE}{kT} + \sinh\frac{pE}{kT}]$$

$$= -pN\,[\frac{kT}{pE} - \coth\frac{pE}{kT}]$$

f) $\frac{pE}{kT} \approx \frac{(1.6 \times 10^{-19})(10^{-9})10^{6}}{(1.38 \times 10^{-23})(300)} \approx .0386$

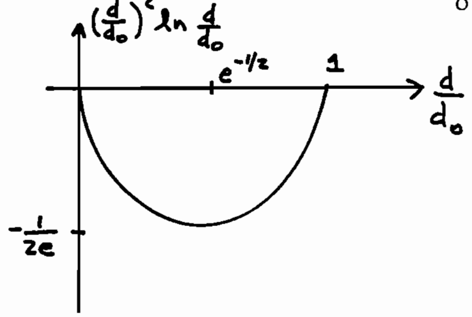
$\coth\frac{pE}{kT} \approx \frac{1}{pE/kT}[1 + \frac{1}{3}(\frac{pE}{kT})^2] \rightarrow P_z \approx \frac{pN}{3}(\frac{pE}{kT})$

g) $P_z = \alpha E \rightarrow \alpha = \frac{p^2 N}{3kT}$

50. From (3.9.3) $f_x = \frac{1}{2}(\varepsilon - \varepsilon_o)\frac{V_o^2 D}{s} = \rho_m ghsD \rightarrow h = \frac{1}{2}\frac{(\varepsilon - \varepsilon_o)V_o^2}{\rho_m gs^2}$

51. (a)-(c) $\sigma_f = \frac{\varepsilon V_o}{d} \rightarrow f_e = \frac{1}{2}\sigma_f E = \frac{1}{2}\frac{\varepsilon V_o^2}{d^2} = -Y\ell n\frac{d}{d_o}$

$$(\frac{d}{d_o})^2 \ell n\frac{d}{d_o} = -\frac{1}{2}\frac{\varepsilon V_o^2}{Y d_o^2}$$



The minimum value of $(\frac{d}{d_o})^2 \ell n\frac{d}{d_o}$ is $-\frac{1}{2e}$ at $\frac{d}{d_o} = e^{-1/2}$. This value is reached when

$V_o^2 = \frac{d_o^2 Y}{\varepsilon e} = \frac{Yd^2}{\varepsilon}$

52. a) $E = E_1$ (in electret); $E = E_2$ (in free space)

$$E_1 d + E_2 s = V_o \qquad E_1 = \left(V_o - \frac{P_o s}{\varepsilon_o}\right)\frac{1}{s+d}$$

$$\longrightarrow$$

$$\varepsilon_o E_1 + P_o = \varepsilon_o E_2 \qquad E_2 = \left(V_o + \frac{P_o d}{\varepsilon_o}\right)\frac{1}{s+d}$$

b) $$f_x = \frac{1}{2}\sigma_f E_2 A = \frac{1}{2}\varepsilon_o E_2^2 A = \frac{1}{2}\frac{\varepsilon_o A}{(s+d)^2}\left[V_o + \frac{P_o d}{\varepsilon_o}\right]^2$$

53. a) $$\nabla\cdot\overline{E} = \frac{dE_x}{dx} = \frac{\rho_o}{\varepsilon} \rightarrow E_x = \frac{\rho_o x}{\varepsilon} + C$$

$$\int_0^s E_x dx = \frac{\rho_o s^2}{2\varepsilon} + Cs = V_o \rightarrow C_1 = \frac{V_o}{s} - \frac{\rho_o s}{2\varepsilon} \rightarrow E_x = \frac{\rho_o}{\varepsilon}\left(x - \frac{s}{2}\right) + \frac{V_o}{s}$$

$$W = \int_0^s \frac{1}{2}\varepsilon E_x^2 A dx = \frac{1}{2}\varepsilon A\int_0^s \left[\frac{\rho_o\left(x - \frac{s}{2}\right)}{\varepsilon} + \frac{V_o}{s}\right]^2 dx$$

$$= \frac{\frac{1}{2}\varepsilon^2 A}{3\rho_o}\left[\frac{\rho_o\left(x-\frac{s}{2}\right)}{\varepsilon} + \frac{V_o}{s}\right]^3\Bigg|_{x=0}^{s}$$

$$= \frac{\varepsilon^2 A}{6\rho_o}\left\{\left[\frac{\rho_o s}{2\varepsilon} + \frac{V_o}{s}\right]^3 - \left[-\frac{\rho_o s}{2\varepsilon} + \frac{V_o}{s}\right]^3\right\}$$

$$= \frac{\varepsilon A s}{6}\left[\left(\frac{\rho_o s}{2\varepsilon}\right)^2 + 3\left(\frac{V_o}{s}\right)^2\right]$$

b) $$\frac{1}{2}CV_o^2 = W \rightarrow C = \frac{2W}{V_o^2} = \frac{\varepsilon A s}{3V_o^2}\left[\left(\frac{\rho_o s}{2\varepsilon}\right)^2 + 3\left(\frac{V_o}{s}\right)^2\right]$$

$$\lim_{\rho_o = 0} C = \frac{\varepsilon A}{s}$$

c) $\sigma_f(x=0) = \varepsilon E_x(x=0)$; $\sigma_f(x=s) = -\varepsilon E_x(x=s)$

$$f_x(x=0) = \frac{1}{2}\sigma_f(x=0)E_x(x=0)A = \frac{1}{2}\varepsilon E_x^2(x=0) = \frac{1}{2}\varepsilon\left[-\frac{\rho_o s}{2\varepsilon} + \frac{V_o}{s}\right]^2 A$$

$$f_x(x=s) = \frac{1}{2}\sigma_f(x=s)E_x(x=s)A = -\frac{1}{2}\varepsilon E_x^2(x=s) = -\frac{1}{2}\varepsilon\left[\frac{\rho_o s}{2\varepsilon} + \frac{V_o}{s}\right]^2 A$$

$$f_{vol} = \int_0^s \rho_o A E_x(x)dx$$

$$= \int_0^s \varepsilon A E_x \frac{dE_x}{dx}\,dx$$

$$= \frac{1}{2}\varepsilon E_x^2 \Big|_{x=0}^{s}$$

$$= \frac{1}{2}\varepsilon E_x^2(x=s) - \frac{1}{2}\varepsilon E_x^2(x=0)$$

d) $f_x(x=0) + f_x(x=s) + f_{vol} = 0$

54. a) $E_r(z=z_o) = \dfrac{V_o}{r\ln\frac{b}{a}}$, $\quad E_r(z=-\infty) = -\dfrac{P_o}{\varepsilon_o}$

$$F_z = P_r \frac{\partial E_z}{\partial r}\ ; \quad \nabla \times \overline{E} = 0 \rightarrow \frac{\partial E_r}{\partial z} = \frac{\partial E_z}{\partial r}$$

$$F_z = P_o \frac{\partial E_r}{\partial z} \rightarrow f_z = \int_{\phi=0}^{2\pi}\int_{r=a}^{b}\int_{z=-\infty}^{z_o} P_o \frac{\partial E_r}{\partial z}\, r dr d\phi dz$$

$$= \int_{r=a}^{b} P_o E_r \Big|_{z=-\infty}^{z_o} 2\pi r dr$$

$$= 2\pi P_o \int_{r=a}^{b} \left[\frac{V_o}{r\ln\frac{b}{a}} + \frac{P_o}{\varepsilon_o}\right] r dr$$

$$= 2\pi P_o \left[\frac{V_o(b-a)}{\ln\frac{b}{a}} + \frac{P_o}{\varepsilon_o}\frac{(b^2-a^2)}{2}\right]$$

b) $C(z) = \dfrac{2\pi}{\ln\frac{b}{a}}\left[\varepsilon z + \varepsilon_o(\ell - z)\right]$

$$f_z = \frac{1}{2}V_o^2\frac{dC}{dz} = \frac{\pi V_o^2}{\ln\frac{b}{a}}(\varepsilon - \varepsilon_o)$$

55. $C(x) = \dfrac{\varepsilon_o(\ell - x)d}{s}$; $\quad f_x = \dfrac{1}{2}V_o^2\dfrac{dC}{dx} = -\dfrac{1}{2}\dfrac{\varepsilon_o d}{s}V_o^2$

56. a) $p = vi = v \frac{d}{dt}[C(\theta)v]$

$$= \frac{d}{dt}\left[\frac{1}{2} C(\theta)v^2\right] + \underbrace{\frac{1}{2} v^2 \frac{dC(\theta)}{d\theta}}_{T} \frac{d\theta}{dt}$$

$$T = \frac{1}{2} v^2 \frac{dC}{d\theta}, \quad W = \frac{1}{2} C(\theta)v^2$$

b) $C(\theta) = \frac{\varepsilon_o 2NR^2}{s}(\theta_o - \theta)$

c) $T = \frac{1}{2} v^2 \frac{dC}{d\theta} = -\frac{NV_o^2 R^2 \varepsilon_o}{s}$

d) $T_m + T_e = 0 \rightarrow -\frac{NV_o^2 R^2 \varepsilon_o}{s} - K(\theta - \theta_s) = 0 \rightarrow \theta - \theta_s = -\frac{NV_o^2 R^2 \varepsilon_o}{Ks}$

At high frequency, the shaft position responds to the time average of $v^2(t)$ (rms value).

e) $C(\theta) = \frac{1}{2}[C_{max} + C_{min}] + \frac{1}{2}[C_{max} - C_{min}]\cos 2\theta$

$$T = \frac{1}{2} v^2 \frac{dC}{d\theta} = -\frac{1}{2} v^2 (C_{max} - C_{min})\sin 2\theta$$

$$= -\frac{1}{2} V_o^2 (C_{max} - C_{min})\cos^2 \omega t \sin 2\theta$$

f) $T = -\frac{1}{2} V_o^2 (C_{max} - C_{min})\left[\frac{1}{2}\sin 2\theta + \frac{1}{4}[\sin(2(\omega t + \theta)) - \sin(2(\omega t - \theta))]\right]$

T has a time average value if $\omega = \pm \omega_m$ $(\theta = \omega_m t + \delta)$.

g) $T_o = -\frac{1}{8} V_o^2 (C_{max} - C_{min})\sin 2\delta$

$$\rightarrow \delta = -\frac{1}{2}\sin^{-1}\left[\frac{8T_o}{V_o^2 (C_{max} - C_{min})}\right]$$

h) $|T_{max}| = \frac{1}{8} V_o^2 (C_{max} - C_{min})$ at $\delta = \pm 45°, \pm 135°$

Section 3.10

57. a) $i = \sigma_f Uw = C\frac{dv}{dt} \rightarrow v(t) = \frac{\sigma_f Uwt}{C}$; $C = 4\pi\varepsilon_o R$

b) Force on the charged belt is

$$F = \sigma_f \frac{v(t)}{\ell} w\ell = \sigma_f w v(t)$$

$$P_{mech} = FU = \sigma_f w v(t) U = \frac{(\sigma_f Uw)^2 t}{C}$$

58. a) $U\frac{\partial \rho_f}{\partial z} + \frac{\sigma}{\varepsilon}\rho_f = 0 \rightarrow \rho_f = \rho_o e^{-\sigma z/\varepsilon U}$

$$\frac{dE}{dz} = \frac{\rho_f}{\varepsilon} \rightarrow E = -\frac{\rho_o U}{\sigma} e^{-\sigma z/\varepsilon U} + C = -\frac{\rho_f U}{\sigma} + C$$

$$i = Jwt = [\underbrace{\rho_f U + \sigma E}_{\sigma C}]wt = \frac{V_o}{R_L} \rightarrow C = \frac{V_o}{\sigma R_L wt}$$

b) $\int_0^\ell E dz = -V_o = \frac{\rho_o \varepsilon U^2}{\sigma^2} e^{-\sigma z/\varepsilon U}\Big|_0^\ell + C\ell$

$$= \frac{\rho_o \varepsilon U^2}{\sigma^2}[e^{-\sigma\ell/\varepsilon U} - 1] + \frac{V_o \ell}{\sigma R_L wt}$$

$$V_o = \frac{\frac{\rho_o \varepsilon U^2}{\sigma^2}[1 - e^{-\sigma\ell/\varepsilon U}]}{1 + \frac{\ell}{\sigma wt R_L}}$$

59. a) $-nC_i v_2 - C\frac{dv_1}{dt} - \frac{v_1}{R} - \frac{(v_1 - v_2)}{R_L} = 0$

$$-nC_i v_1 - C\frac{dv_2}{dt} - \frac{v_2}{R} + \frac{(v_1 - v_2)}{R_L} = 0$$

$$v_1 = V_1 e^{st}, \quad v_2 = V_2 e^{st}$$

$$V_1[Cs + \frac{1}{R} + \frac{1}{R_L}] + V_2[nC_i - \frac{1}{R_L}] = 0$$

$$V_1[nC_i - \frac{1}{R_L}] + V_2[Cs + \frac{1}{R} + \frac{1}{R_L}] = 0$$

$$[Cs + \frac{1}{R} + \frac{1}{R_L}]^2 = [nC_i - \frac{1}{R_L}]^2$$

$$s = -\frac{1}{C}\,[\frac{1}{R} + \frac{1}{R_L} \pm \frac{1}{R_L} \mp nC_i]$$

$$s_1 = -\frac{1}{C}\,[\frac{1}{R} + \frac{2}{R_L} - nC_i], \quad s_2 = -\frac{1}{C}\,[\frac{1}{R} + nC_i]$$

$$s_1 > 0 \rightarrow nC_i > \frac{1}{R} + \frac{2}{R_L} \quad \text{(self-excited)}$$

b) $n = 10,\ C_i = 2\text{pf},\ C = 10\ \text{pf},\ R_L = R$

$$R_{min} = \frac{3}{nC_i} = 1.5 \times 10^{11}\ \Omega$$

c) $nC_i v_1 + C\frac{dv_2}{dt} + \frac{v_2}{R} = 0 \ ; \quad v_1 = V_1 e^{st}$

$$nC_i v_2 + C\frac{dv_3}{dt} + \frac{v_3}{R} = 0 \ ; \quad v_2 = V_2 e^{st}$$

$$nC_i v_3 + C\frac{dv_1}{dt} + \frac{v_1}{R} = 0 \ ; \quad v_3 = V_3 e^{st}$$

$$\begin{bmatrix} nC_i & Cs + \frac{1}{R} & 0 \\ 0 & nC_i & Cs + \frac{1}{R} \\ Cs + \frac{1}{R} & 0 & nCi \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$$

$$(nC_i)^3 + (Cs + \frac{1}{R})^3 = 0 \rightarrow s = -\frac{1}{RC} - e^{j2\pi r/3}\,\frac{nC_i}{C}, \quad r = 1,\ 2,\ 3$$

For self-excitation

$$\frac{nC_i}{2} > \frac{1}{R} \quad \text{with} \quad \omega_o = \frac{\sqrt{3}}{2}\frac{nC_i}{C}$$

d) $nC_i V_k + (Cs + \frac{1}{R})V_{k+1} = 0 \; ; \qquad V_k = A\lambda^k$

$$[Cs + \frac{1}{R}]\lambda + nC_i = 0 \rightarrow \lambda = -\frac{nC_i}{Cs + \frac{1}{R}}$$

$$\lambda^{N+1} = \lambda \rightarrow \lambda^N = 1 \rightarrow -\frac{nC_i}{Cs + \frac{1}{R}} = e^{j2\pi r/N}, \qquad r = 1, 2, \ldots N$$

$$s = -\frac{nC_i}{C} e^{-j2\pi r/N} - \frac{1}{RC}$$

For self-excitation

$$\mathrm{Re}(s) > 0 \rightarrow \cos 2\pi r/N > -\frac{1}{nRC_i}, \quad \omega_o = |\mathrm{Im}(s)| = \left|\frac{nC_i}{C} \sin \frac{2\pi r}{N}\right|$$

CHAPTER 4

ELECTRIC FIELD BOUNDARY VALUE PROBLEMS

Section 4.1

1. a) $\overline{E} = -\frac{V_o}{ab}[y\overline{i}_x + x\overline{i}_y]$; $\sigma_f(y=0) = \varepsilon_o E_y(y=0) = -\frac{V_o x}{ab}$

$$\sigma_f(x=0) = \varepsilon_o E_x(x=0) = -\frac{V_o y}{ab}$$

$$q_T(x=0) = D\int_0^{y_o} \sigma_f(x=0)dy = -\frac{V_o D y_o^2}{2ab}$$

$$q_T(y=0) = D\int_0^{x_o} \sigma_f(y=0)dx = -\frac{V_o D x_o^2}{2ab}$$

$$q_T = q_T(x=0) + q_T(y=0) = -\frac{V_o D}{2ab}(x_o^2 + y_o^2)$$

$$C = \left|\frac{q_T}{V_o}\right| = \frac{D}{2ab}(x_o^2 + y_o^2)$$

b) $\frac{1}{2}mv^2 + qV = \text{constant} = qV_o \rightarrow v = \sqrt{\frac{2q}{m}(V_o - V)}$

$$= \sqrt{\frac{2qV_o}{m}\left(1 - \frac{xy}{ab}\right)}$$

c) $v_{impact} = \sqrt{\frac{2qV_o}{m}}$

2. a)
$$V = \begin{cases} [A_1 \sin ay + A_2 \cos ay]e^{-ax} & x > 0 \\ [B_1 \sin ay + B_2 \cos ay]e^{+ax} & x < 0 \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} ae^{-ax}\{[A_1 \sin ay + A_2 \cos ay]\overline{i}_x - [A_1 \cos ay - A_2 \sin ay]\overline{i}_y\} & x > 0 \\ -ae^{+ax}\{[B_1 \sin ay + B_2 \cos ay]\overline{i}_x + [B_1 \cos ay - B_2 \sin ay]\overline{i}_y\} & x < 0 \end{cases}$$

Boundary conditions:

$$E_y(x=0_+) = E_y(x=0_-) \rightarrow A_1 = B_1,\ A_2 = B_2$$

$$\varepsilon[E_x(x=0_+) - E_x(x=0_-)] = \sigma_f = \sigma_o \cos ay \rightarrow A_1 = B_1 = 0,\ A_2 = B_2 = \frac{\sigma_o}{2a\varepsilon}$$

$$V = \begin{cases} \dfrac{\sigma_o}{2a\varepsilon}\cos ay\, e^{-ax} & x > 0 \\ \dfrac{\sigma_o}{2a\varepsilon}\cos ay\, e^{+ax} & x < 0 \end{cases}, \quad \overline{E} = -\nabla V = \begin{cases} \dfrac{\sigma_o e^{-ax}}{2\varepsilon}[\cos ay\,\overline{i}_x + \sin ay\,\overline{i}_y] & x > 0 \\ \dfrac{-\sigma_o e^{ax}}{2\varepsilon}[\cos ay\,\overline{i}_x - \sin ay\,\overline{i}_y] & x < 0 \end{cases}$$

b) $$\frac{dy}{dx} = \frac{E_y}{E_x} = \pm \tan ay \quad \begin{matrix} x > 0 \\ x < 0 \end{matrix} \rightarrow e^{\mp ax} \sin ay = \text{constant}$$

3. a) $$V = \begin{cases} V_o \cos ay\, e^{a(x + \frac{d}{2})} & x < -d/2 \\ -V_o \cos ay \dfrac{\sinh ax}{\sinh \frac{ad}{2}} & |x| < d/2 \\ -V_o \cos ay\, e^{-a(x - \frac{d}{2})} & x > d/2 \end{cases}$$

$$\overline{E} = -\nabla V \begin{cases} -V_o a e^{a(x + \frac{d}{2})}[\cos ay\,\overline{i}_x - \sin ay\,\overline{i}_y] & x < -d/2 \\ \dfrac{V_o a}{\sinh \frac{ad}{2}}[\cosh ax \cos ay\,\overline{i}_x - \sinh ax \sin ay\,\overline{i}_y] & |x| < d/2 \\ -V_o a\, e^{-a(x - \frac{d}{2})}[\cos ay\,\overline{i}_x + \sin ay\,\overline{i}_y] & x > d/2 \end{cases}$$

b) $$\sigma_f(x=-\tfrac{d}{2}) = \varepsilon[E_x(x=-\tfrac{d_+}{2}) - E_x(x=-\tfrac{d_-}{2})]$$

$$= \varepsilon V_o a \cos ay\, [\coth \tfrac{ad}{2} + 1]$$

$$\sigma_f(x=\tfrac{d}{2}) = \varepsilon[E_x(x=\tfrac{d_+}{2}) - E_x(x=\tfrac{d_-}{2})]$$

$$= -\varepsilon V_o a \cos ay\, [1 + \coth \tfrac{ad}{2}]$$

4. a) $$V(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sin \frac{n\pi y}{d} [A_n \cosh \frac{n\pi x}{d} + B_n \sinh \frac{n\pi x}{d}]$$

$$V(x=0) = V_1 = \frac{4V_1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d}}{n} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sin \frac{n\pi y}{d} A_n \rightarrow A_n = \frac{4V_1}{n\pi}$$

$$V(x=\ell) = 0 = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sin \frac{n\pi y}{d} \left[A_n \cosh \frac{n\pi\ell}{d} + B_n \sinh \frac{n\pi\ell}{d}\right]$$

$$B_n = -A_n \coth \frac{n\pi\ell}{d}$$

$$V(x,y) = \frac{4V_1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d}}{n} \left[\cosh \frac{n\pi x}{d} - \coth \frac{n\pi\ell}{d} \sinh \frac{n\pi x}{d}\right]$$

$$= \frac{4V_1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d}}{n \sinh \frac{n\pi\ell}{d}} \sinh \frac{n\pi(\ell - x)}{d}$$

$$\overline{E} = -\nabla V = -\frac{4V_1}{d} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{\sinh \frac{n\pi\ell}{d}} \left[\cos \frac{n\pi y}{d} \sinh \frac{n\pi(\ell-x)}{d} \overline{i}_y - \sin \frac{n\pi y}{d} \cosh \frac{n\pi(\ell-x)}{d} \overline{i}_x\right]$$

b) $$V = \frac{4V_1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d}}{n \sinh \frac{n\pi\ell}{d}} \sinh \frac{n\pi(\ell-x)}{d} + \frac{4V_2}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi x}{\ell} \sinh \frac{n\pi(d-y)}{\ell}}{n \sinh \frac{n\pi d}{\ell}}$$

c) $$V = \frac{4V_1}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d} \sinh \frac{n\pi(\ell-x)}{d}}{n \sinh \frac{n\pi\ell}{d}} + \frac{4V_2}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi x}{\ell} \sinh \frac{n\pi(d-y)}{\ell}}{n \sinh \frac{n\pi d}{\ell}}$$

$$+ \frac{4V_3}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi y}{d} \sinh \frac{n\pi x}{d}}{n \sinh \frac{n\pi\ell}{d}} + \frac{4V_4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin \frac{n\pi x}{\ell} \sinh \frac{n\pi y}{\ell}}{n \sinh \frac{n\pi d}{\ell}}$$

5. a) $c = \sqrt{a^2 + b^2}$

$$V(x,y,z) = \begin{cases} \dfrac{V_o \sin ax \cos bz \sinh c(y - d)}{\sinh c(s - d)} & s \leq y \leq d \\ \\ \dfrac{V_o \sin ax \cos bz \sinh cy}{\sinh cs} & 0 \leq y \leq s \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} \dfrac{-V_o}{\sinh c(s-d)} [a \cos ax \cos bz \sinh c(y - d)\overline{i}_x & s < y < d \\ \quad -b \sin ax \sin bz \sinh c(y - d)\overline{i}_z & \\ \quad +c \sin ax \cos bz \cosh c(y - d)\overline{i}_y] & \\ \\ \dfrac{-V_o}{\sinh cs} [a \cos ax \cos bz \sinh cy\ \overline{i}_x - b \sin ax \sin bz \sinh cy\ \overline{i}_z & \\ \quad + c \sin ax \cos bz \cosh cy\ \overline{i}_y] & 0 < y < s \end{cases}$$

b) $\sigma_f(y=0) = \epsilon E_y(y=0) = \dfrac{-\epsilon V_o c}{\sinh cs} \sin ax \cos bz$

$$\sigma_f(y=d) = -\epsilon E_y(y=d) = \frac{\epsilon V_o c}{\sinh c(s-d)} \sin ax \cos bz$$

$$\sigma_f(y=s) = \epsilon[E_y(y=s_+) - E_y(y=s_-)]$$

$$= -\epsilon V_o c \sin ax \cos bz[\coth c(s - d) - \coth cs]$$

6. $\sigma_f = \dfrac{4\sigma_o}{\pi} \sum_{n\ \text{odd}} \dfrac{1}{n} \sin \dfrac{n\pi y}{d}$

a)
$$V = \begin{cases} \sum_{n\ \text{odd}} V_n \sin \dfrac{n\pi y}{d}\, e^{-n\pi x/d} & x \geq 0 \\ \\ \sum_{n\ \text{odd}} V_n \sin \dfrac{n\pi y}{d}\, e^{+n\pi x/d} & x \leq 0 \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} -\sum_{n\ \text{odd}} V_n \dfrac{n\pi}{d} [\cos \dfrac{n\pi y}{d}\, \overline{i}_y - \sin \dfrac{n\pi y}{d}\, \overline{i}_x] e^{-n\pi x/d} & x > 0 \\ \\ -\sum_{n\ \text{odd}} V_n \dfrac{n\pi}{d} [\cos \dfrac{n\pi y}{d}\, \overline{i}_y + \sin \dfrac{n\pi y}{d}\, \overline{i}_x]\, e^{+n\pi x/d} & x < 0 \end{cases}$$

$$\sigma_f = \varepsilon[E_x(x=0_+) - E_x(x=0_-)] \to \frac{4\sigma_o}{n\pi} = 2\varepsilon V_n \frac{n\pi}{d}$$

$$V_n = \frac{2\sigma_o d}{\varepsilon(n\pi)^2} \qquad n = 1, 3, 5, \ldots$$

b) $$\sigma_f(y=0) = \varepsilon E_y(y=0) = \begin{cases} -\sum\limits_{n\ \mathrm{odd}} \dfrac{2\sigma_o}{(n\pi)} e^{-n\pi x/d} & x > 0 \\ -\sum\limits_{n\ \mathrm{odd}} \dfrac{2\sigma_o}{(n\pi)} e^{+n\pi x/d} & x < 0 \end{cases}$$

$$\sigma_f(y=d) = -\varepsilon E_y(y=d) = \sigma_f(y=0)$$

c) $$q_T(y=0) = q_T(y=d) = 2\int_0^\infty \sigma_f(y=0)\,dx$$

$$= -4\sigma_o \sum_{n\ \mathrm{odd}} \frac{1}{n\pi} \int_0^\infty e^{-(n\pi x/d)}\,dx$$

$$= 4\sigma_o \sum_{n\ \mathrm{odd}} \frac{d}{(n\pi)^2} e^{-n\pi x/d}\Big|_{x=0}^{\infty}$$

$$= -4\sigma_o d \underbrace{\sum_{n\ \mathrm{odd}} \left(\frac{1}{n\pi}\right)^2}_{1/8} = -\frac{\sigma_o d}{2}$$

7. a) $$\nabla^2 V_p = \frac{d^2 V_p}{dx^2} = -\frac{\rho_o}{\varepsilon_o}\sin ax \to V_p = \frac{\rho_o}{\varepsilon_o a^2}\sin ax$$

$V_p(y=0) \neq 0$ (Boundary condition not satisfied)

b) $$\nabla^2 V = \frac{-\rho_o}{\varepsilon_o}\sin ax \to V = \begin{cases} A\sin ax\, e^{-a(y-d)} & y \geq d \\ [B\sinh ay + C\cosh ay]\sin ax + \dfrac{\rho_o}{\varepsilon_o a^2}\sin ax & 0 \leq y \leq d \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} -Aa[\cos ax\, \overline{i}_x - \sin ax\, \overline{i}_y]e^{-a(y-d)} & y \geq d \\ -\dfrac{\rho_o}{\varepsilon_o a}\cos ax\overline{i}_x - a\{\cos ax[B\sinh ay + C\cosh ay]\overline{i}_x \\ \quad + \sin ax[B\cosh ay + C\sinh ay]\overline{i}_y\} & 0 < y \leq d \end{cases}$$

Boundary Conditions

$$V(y=0) = 0 \rightarrow \frac{\rho_o}{\varepsilon_o a^2} + C = 0$$

$$V(y=d_-) = V(y=d_+) \rightarrow A = \frac{\rho_o}{\varepsilon_o a^2} + B\sinh ad + C\cosh ad$$

$$\sigma_f(y=d) = \varepsilon_o[E_y(y=d_+) - E_y(y=d_-)] = 0 \rightarrow B\cosh ad + C\sinh ad = -A$$

$$C = \frac{-\rho_o}{\varepsilon_o a^2}, \quad A = \frac{\rho_o}{\varepsilon_o a^2}\frac{(1-e^{-ad})^2}{2}, \quad B = \frac{\rho_o}{\varepsilon_o a^2}[1 - e^{-ad}]$$

c) $$\sigma_f(y=0) = \varepsilon_o E_y(y=0) = -\varepsilon_o a\sin ax B = -\frac{\rho_o}{a}(1 - e^{-ad})\sin ax$$

d) $$F_y = \frac{1}{2}\sigma_f(y=0)E_y(y=0) = \frac{1}{2}\frac{\sigma_f(y=0)^2}{\varepsilon_o} = \frac{1}{2}\frac{\rho_o^2}{\varepsilon_o a^2}(1 - e^{-ad})^2\sin^2 ax$$

$$\begin{aligned} f_y &= \int_0^{2\pi/a} F_y dx \\ &= \frac{1}{2}\frac{\rho_o^2}{\varepsilon_o a^2}(1 - e^{-ad})^2\int_0^{2\pi/a}\sin^2 ax\,dx \\ &= \frac{\pi}{2}\frac{\rho_o^2}{\varepsilon_o a^3}(1 - e^{-ad})^2 \end{aligned}$$

$$\overline{F}_v = \rho_f\overline{E} = \rho_f[E_x\overline{i}_x + E_y\overline{i}_y]$$

$$f_{vx} = \int_{x=0}^{2\pi/a}\int_{y=0}^{d}\rho_f E_x dxdy = -\rho_o a\int_{x=0}^{2\pi/a}\int_{y=0}^{d}[B\sinh ay + C\cosh ay + \frac{\rho_o}{\varepsilon_o a^2}]\sin ax\cos ax\,dxdy$$

$$= 0$$

when integrated over x$\left[\int_0^{2\pi/a} \sin ax \cos ax\, dx = 0\right]$.

$$f_{vy} = \int_{x=0}^{2\pi/a} \int_{y=0}^{d} \rho_f E_y dxdy$$

$$= -\rho_o a \int_{x=0}^{2\pi/a} \int_{y=0}^{d} \sin^2 ax\{B\cosh ay + C\sinh ay\}dxdy$$

$$= -\frac{\rho_o \pi}{a}[B\sinh ay + C\cosh ay]\Big|_{y=0}^{d}$$

$$= -\frac{\rho_o \pi}{a}[B\sinh ad + C(\cosh ad - 1)]$$

$$= -\frac{\rho_o^2 \pi}{2\varepsilon_o a^3}(1 - e^{-ad})^2 = -f_y$$

The total force on the system is zero ($f_y + f_{vy} = 0$).

e) $\nabla\cdot\overline{D} = 0 = \nabla\cdot[\varepsilon_o\overline{E} + \overline{P}]$

$$\nabla\cdot\overline{P} = \frac{\partial P_y}{\partial y} = 0 \rightarrow \nabla\cdot\overline{E} = 0 \rightarrow \nabla^2 V = 0$$

$$V = \begin{cases} A\sin ax\, e^{-a(y-d)} & y \geq d \\ [B\sinh ay + C\cosh ay]\sin ax & 0 \leq y \leq d \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} -Aa[\cos ax\overline{i}_x - \sin ax\overline{i}_y]e^{-a(y-d)} & y \geq d \\ -a\{\cos ax[B\sinh ay + C\cosh ay]\overline{i}_x \\ \quad + \sin ax[B\cosh ay + C\sinh ay]\overline{i}_y\} & 0 < y < d \end{cases}$$

<u>Boundary Conditions</u>:

$V(y=0) = 0 \rightarrow C = 0$

$V(y=d_+) = V(y=d_-) \rightarrow A = B\sinh ad + C\cosh ad$

$$D_y(y=d_+) = D_y(y=d_-) \rightarrow \varepsilon_o E_y(y=d_+) = \varepsilon_o E_y(y=d_-) + P_o \sin ax$$

$$A = \frac{P_o}{\varepsilon_o a} e^{-ad} \sinh ad, \quad B = \frac{P_o}{\varepsilon_o a} e^{-ad}, \quad C = 0$$

$$\sigma_f(y=0) = \varepsilon_o E_y(y=0) = -a\varepsilon_o B \sin ax = -P_o e^{-ad} \sin ax$$

8. a) i) $z^2 = (x + jy)^2 = (x^2 - y^2) + 2jxy \rightarrow u = x^2 - y^2, \quad v = 2xy$

ii) $\sin z = \sin(x + jy) = \sin x \cos jy + \cos x \sin jy$

$= \sin x \cosh y + j\cos x \sinh y \rightarrow u = \sin x \cosh y, v = \cos x \sinh y$

iii) $\cos z = \cos(x + jy) = \cos x \cos jy - \sin x \sin jy$

$= \cos x \cosh y - j\sin x \sinh y \rightarrow u = \cos x \cosh y, \quad v = -\sin x \sinh y$

iv) $e^z = e^{(x+jy)} = e^x[\cos y + j\sin y] \rightarrow u = e^x \cos y, \quad v = e^x \sin y$

v) $\ell n z = \ell n(x + jy) = \ell n\sqrt{x^2+y^2} + j\tan^{-1}\frac{y}{x} \rightarrow u = \ell n\sqrt{x^2+y^2}, \quad v = \tan^{-1}\frac{y}{x}$

b) $$\frac{dw}{dz} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{1}{j}\left[\frac{\partial u}{\partial y} + j\frac{\partial v}{\partial y}\right]$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

c) $$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right] \rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right] \rightarrow -\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right] \rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right] \rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

d) $\nabla u = \frac{\partial u}{\partial x}\bar{i}_x + \frac{\partial u}{\partial y}\bar{i}_y$ is perpendicular to lines of constant u

$\nabla v = \frac{\partial v}{\partial x}\bar{i}_x + \frac{\partial v}{\partial y}\bar{i}_y$ is perpendicular to lines of constant v

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0$$

$\nabla u \perp \nabla v \rightarrow$ lines of constant u and v are perpendicular.

Section 4.3

9. a)

$$V(\phi) = \begin{cases} A_1\phi + B_1 & 0 < \phi < \alpha \\ A_2\phi + B_2 & \alpha < \phi < \pi \end{cases}$$

$$\bar{E} = -\nabla V = \begin{cases} -\frac{A_1}{r}\bar{i}_\phi & 0 < \phi < \alpha \\ -\frac{A_2}{r}\bar{i}_\phi & \alpha < \phi < \pi \end{cases}$$

Boundary conditions:

$$V(\phi=0) = V_o = B_1$$

$$V(\phi=\pi) = 0 = A_2\pi + B_2$$

$$V(\phi=\alpha_-) = V(\phi=\alpha_+) \rightarrow A_1\alpha + B_1 = A_2\alpha + B_2$$

At $t = 0$

$$\sigma_f(\phi=\alpha) = \varepsilon_2 E_\phi(\phi=\alpha_+) - \varepsilon_1 E_\phi(\phi=\alpha_-) = 0 \rightarrow \varepsilon_2 A_2 = \varepsilon_1 A_1$$

$$A_1 = \frac{\varepsilon_2}{\varepsilon_1}A_2 = \frac{-\varepsilon_2 V_o}{\varepsilon_2\alpha + \varepsilon_1(\pi - \alpha)}; \quad B_2 = -A_2\pi, \quad B_1 = V_o$$

At $t = \infty$

$$J_\phi(\phi=\alpha_+) = J_\phi(\phi=\alpha_-) \rightarrow \sigma_2 E_\phi(\phi=\alpha_+) = \sigma_1 E_\phi(\phi=\alpha_-) \rightarrow \sigma_2 A_2 = \sigma_1 A_1$$

$$A_1 = \frac{\sigma_2}{\sigma_1} A_2 = \frac{-\sigma_2 V_o}{\sigma_2\alpha + \sigma_1(\pi - \alpha)}, \quad B_2 = -A_2\pi, \; B_1 = V_o$$

<u>Transient Interval</u>

$$J_\phi(\phi=\alpha_+) - J_\phi(\phi=\alpha_-) + \frac{\partial}{\partial t}\sigma_f(\phi=\alpha) = 0$$

$$\sigma_2 E_\phi(\phi=\alpha_+) - \sigma_1 E_\phi(\phi=\alpha_-) + \frac{\partial}{\partial t}[\varepsilon_2 E_\phi(\phi=\alpha_+) - \varepsilon_1 E_\phi(\phi=\alpha_-)] = 0$$

$$\frac{\partial A_1}{\partial t} + \frac{A_1}{\tau} = -\frac{\sigma_2 V_o}{\varepsilon_2\alpha + \varepsilon_1(\pi - \alpha)}; \qquad \tau = \frac{\varepsilon_2\alpha + \varepsilon_1(\pi - \alpha)}{\sigma_2\alpha + \sigma_1(\pi - \alpha)}$$

$$A_1 = -\frac{\varepsilon_2 V_o e^{-t/\tau}}{\varepsilon_2\alpha + \varepsilon_1(\pi - \alpha)} - \frac{\sigma_2 V_o(1 - e^{-t/\tau})}{\sigma_2\alpha + \sigma_1(\pi - \alpha)}$$

$$A_2 = \frac{A_1\alpha + V_o}{\alpha - \pi}, \quad B_1 = V_o, \quad B_2 = -A_2\alpha$$

b) $\sigma_f(\phi=\alpha) = \varepsilon_2 E_\phi(\phi=\alpha_+) - \varepsilon_1 E_\phi(\phi=\alpha_-)$

$$= \frac{-\varepsilon_2 A_2 + \varepsilon_1 A_1}{r} = \frac{(\varepsilon_2 - \sigma_2\tau)V_o(1 - e^{-t/\tau})}{(\pi - \alpha)r}$$

$$= \frac{(\varepsilon_2\sigma_1 - \varepsilon_1\sigma_2)V_o(1 - e^{-t/\tau})}{[\sigma_2\alpha + \sigma_1(\pi - \alpha)]r}$$

c) $\sigma_f(\phi=0) = \varepsilon_1 E_\phi(\phi=0) = -\dfrac{\varepsilon_1 A_1}{r}$

$$q(\phi=0) = \int_{r=a}^{b} \sigma_f \ell dr = -\varepsilon_1 A_1 \ell \ln\frac{b}{a}$$

$$C = \frac{q(\phi=0,\ t=0)}{V_o} = -\frac{\varepsilon_1\ell \ln\frac{b}{a} A_1(t=0)}{V_o} = \frac{\varepsilon_1\varepsilon_2\ell \ln\frac{b}{a}}{\varepsilon_2\alpha + \varepsilon_1(\pi - \alpha)}$$

$$R = \frac{V_o}{I(t=\infty)}\,,\quad I(t=\infty) = \int_{r=0}^{a} J_r(t=\infty)\,\ell dr$$

$$= \int_{r=0}^{a} -\frac{\sigma_1 A_1(t=\infty)\,\ell dr}{r}$$

$$= -\,\sigma_1 \ell\, A_1(t=\infty)\,\ell n\,\frac{b}{a} = \frac{\sigma_1\sigma_2 \ell\ \ell n\,\frac{b}{a}\,V_o}{\sigma_2\alpha + \sigma_1(\pi - \alpha)}$$

$$R = \frac{V_o}{I(t=\infty)} = \frac{\sigma_2\alpha + \sigma_1(\pi - \alpha)}{\sigma_1\sigma_2\ell\ \ell n\,\frac{b}{a}}$$

10. a)

$$V(r,\phi) = \begin{cases} V_o(r/a)^n \sin n\phi & 0 < r < a \\ V_o(r/a)^{-n} \sin n\phi & r > a \end{cases}$$

$$\overline{E} = -\nabla V = -\frac{\partial V}{\partial r}\,\overline{i}_r - \frac{1}{r}\frac{\partial V}{\partial \phi}\,\overline{i}_\phi$$

$$= \begin{cases} -\frac{V_o n}{a}\left\{\left(\frac{r}{a}\right)^{n-1}\left[\sin n\phi\,\overline{i}_r + \cos n\phi\,\overline{i}_\phi\right]\right\} & 0 < r < a \\ \frac{V_o n}{a}\left\{\left(\frac{r}{a}\right)^{-n-1}\left[\sin n\phi\,\overline{i}_r - \cos n\phi\,\overline{i}_\phi\right]\right\} & r > a \end{cases}$$

b) $$V(r=a,\phi) = \sum_{n=1}^{\infty} V_n \sin n\phi \rightarrow V_n = \frac{2V_o}{n\pi} \qquad n \text{ odd}$$

c)

$$V(r,\phi) = \begin{cases} \sum\limits_{n \text{ odd}} \frac{2V_o}{n\pi}\left(\frac{r}{a}\right)^n \sin n\phi & 0 < r < a \\ \sum\limits_{n \text{ odd}} \frac{2V_o}{n\pi}\left(\frac{r}{a}\right)^{-n} \sin n\phi & r > a \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} \sum\limits_{n \text{ odd}} -\frac{2V_o n}{a}\left\{\left(\frac{r}{a}\right)^{n-1}\left[\sin n\phi\,\overline{i}_r + \cos n\phi\,\overline{i}_\phi\right]\right\} & 0 < r < a \\ \sum\limits_{n \text{ odd}} \frac{2V_o n}{a}\left\{\left(\frac{r}{a}\right)^{-n-1}\left[\sin n\phi\,\overline{i}_r - \cos n\phi\,\overline{i}_\phi\right]\right\} & r > a \end{cases}$$

11.
$$V(r,\phi) = \begin{cases} A_1 r\cos\phi & 0 \le r \le a \\ (A_2 r + \dfrac{B_2}{r})\cos\phi & a \le r \le b \\ (A_3 r + \dfrac{B_3}{r})\cos\phi & r \ge b \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} -A_1[\overline{i}_r\cos\phi - \overline{i}_\phi\sin\phi] & 0 < r < a \\ -(A_2 - \dfrac{B_2}{r^2})\cos\phi\overline{i}_r + (A_2 + \dfrac{B_2}{r^2})\sin\phi\overline{i}_\phi & a < r < b \\ -(A_3 - \dfrac{B_3}{r^2})\cos\phi\overline{i}_r + (A_3 + \dfrac{B_3}{r^2})\sin\phi\overline{i}_\phi & r > b \end{cases}$$

<u>Boundary conditions</u>:

$$V(r=a_-) = V(r=a_+) \rightarrow A_1 a = A_2 a + \frac{B_2}{a}$$

$$V(r=b_-) = V(r=b_+) \rightarrow A_2 b + \frac{B_2}{b} = A_3 b + \frac{B_3}{b}$$

$$\varepsilon E_r(r=a_+) = \varepsilon_o E_r(r=a_-) \rightarrow \varepsilon(A_2 - \frac{B_2}{a^2}) = \varepsilon_o A_1$$

$$\varepsilon_o E_r(r=b_+) = \varepsilon E_r(r=b_-) \rightarrow \varepsilon(A_2 - \frac{B_2}{b^2}) = \varepsilon_o(A_3 - \frac{B_3}{b^2})$$

$$\overline{E}(r\rightarrow\infty) = E_o\overline{i}_x = E_o[\overline{i}_r\cos\phi - \overline{i}_\phi\sin\phi] \rightarrow A_3 = -E_o$$

$$A_1 = \frac{-4E_o}{2(1 + \frac{a^2}{b^2}) + (\frac{\varepsilon}{\varepsilon_o} + \frac{\varepsilon_o}{\varepsilon})(1 - \frac{a^2}{b^2})}, \quad A_2 = \frac{A_1}{2}(1 + \frac{\varepsilon_o}{\varepsilon}), \quad B_2 = \frac{a^2 A_1}{2}(1 - \frac{\varepsilon_o}{\varepsilon})$$

$$A_3 = -E_o, \quad B_3 = -E_o - \frac{\varepsilon}{\varepsilon_o}\frac{A_1}{2}[1 + \frac{\varepsilon_o}{\varepsilon} - \frac{a^2}{b^2}]$$

12.
$$V(r,\phi) = \begin{cases} Ar\cos\phi & 0 \le r \le a \\ (Br + \dfrac{C}{r})\cos\phi & r \ge a \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} -A(\cos\phi\,\overline{i}_r - \sin\phi\,\overline{i}_\phi) & 0 \le r < a \\ -(B - \frac{C}{r^2})\cos\phi\,\overline{i}_r + (B + \frac{C}{r^2})\sin\phi\,\overline{i}_\phi & r > a \end{cases}$$

Boundary conditions:

$$V(r=a_-) = V(r=a_+) \rightarrow Aa = Ba + \frac{C}{a}$$

$$\overline{E}(r\rightarrow\infty) = E_o(\overline{i}_r\cos\phi - \overline{i}_\phi\sin\phi) \rightarrow B = -E_o$$

$$\varepsilon_o E_r(r=a_+) + P_1\cos\phi = \varepsilon_o E_r(r=a_-) + P_2\cos\phi$$

$$V(r,\phi) = \begin{cases} [\frac{P_2 - P_1}{2\varepsilon_o} - E_o]r\cos\phi & 0 \le r \le a \\ [-E_o r + \frac{(P_2 - P_1)a^2}{2\varepsilon_o r}]\cos\phi & r > a \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} -[\frac{P_2 - P_1}{2\varepsilon_o} - E_o][\cos\phi\,\overline{i}_r - \sin\phi\,\overline{i}_\phi] & 0 \le r < a \\ [E_o + \frac{a^2(P_2 - P_1)}{2\varepsilon_o r^2}]\cos\phi\,\overline{i}_r - [E_o - \frac{a^2(P_2-P_1)}{2\varepsilon_o r^2}]\sin\phi\,\overline{i}_\phi & r > a \end{cases}$$

13. a) $\overline{E} = [E_o(1 + \frac{a^2}{r^2})\cos\phi + \frac{\lambda(t)}{2\pi\varepsilon r}]\ \overline{i}_r - E_o(1 - \frac{a^2}{r^2})\sin\phi\,\overline{i}_\phi$

$$\overline{E} = 0 \rightarrow E_o(1 + \frac{a^2}{r^2})\cos\phi + \frac{\lambda(t)}{2\pi\varepsilon r} = 0$$

$$E_o(1 - \frac{a^2}{r^2})\sin\phi = 0 \rightarrow \phi = 0,\pi$$
$$r = a$$

At $r = a$

$$2E_o\cos\phi + \frac{\lambda(t)}{2\pi\varepsilon a} \rightarrow \cos\phi_c = \frac{-\lambda(t)}{4\pi\varepsilon E_o a}$$

At $\phi=0,\pi$

$$\pm E_o(1+\frac{a^2}{r^2}) + \frac{\lambda(t)}{2\pi\varepsilon r} = 0 \rightarrow r^2 + a^2 \pm \frac{\lambda(t)r}{2\pi\varepsilon E_o} = 0$$

$$r = \mp \frac{\lambda(t)}{4\pi\varepsilon E_o} \pm \sqrt{(\frac{\lambda}{4\pi\varepsilon E_o})^2 - a^2}$$

$$\frac{1}{r}\frac{\partial\Sigma}{\partial\phi} = E_r = E_o(1 + \frac{a^2}{r^2})\cos\phi + \frac{\lambda(t)}{2\pi\varepsilon r}$$

$$\rightarrow \frac{\Sigma}{E_o a} = \frac{(\frac{r}{a})^2+1}{(\frac{r}{a})}\sin\phi + \frac{\lambda}{2\pi\varepsilon E_o a}\phi = \text{constant}$$

$$-\frac{\partial\Sigma}{\partial r} = E_\phi = -E_o(1 - \frac{a^2}{r^2})\sin\phi$$

b) $E_r(r=a) < 0 \rightarrow 2E_o\cos\phi + \frac{\lambda(t)}{2\pi\varepsilon a} < 0 \rightarrow \cos\phi < \frac{-\lambda(t)}{4\pi\varepsilon a E_o}$

c) $-1 \le \cos\phi \le 1 \rightarrow \lambda_{max} = 4\pi\varepsilon a E_o$

d) $dI = -q_o n_o \mu E_r(r=a)a\,d\phi$

$$I = -q_o n_o \mu a \int_{\phi_c}^{2\pi-\phi_c} [2E_o\cos\phi + \frac{\lambda}{2\pi\varepsilon a}]d\phi; \quad \phi_c = \cos^{-1}\frac{\lambda}{4\pi\varepsilon a E_o}$$

$$= -q_o n_o \mu a[2E_o\sin\phi + \frac{\lambda\phi}{2\pi\varepsilon a}]\Bigg|_{\phi_c}^{2\pi-\phi_c}$$

$$= -q_o n_o \mu a[-4E_o\sin\phi_c + \frac{\lambda}{\pi\varepsilon a}(\pi - \phi_c)]$$

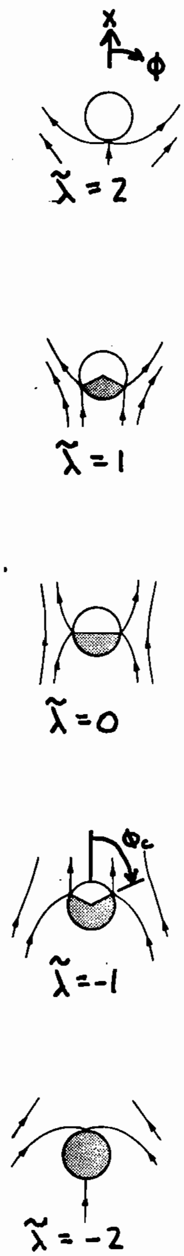
e) $\lim\limits_{\substack{r\to\infty \\ \phi\to\pi}} y^* = r\sin\phi = \frac{\Sigma}{E_o} - \frac{\lambda\pi}{2\pi\varepsilon E_o}$

$$\frac{\Sigma}{E_o} = 2a\sin\phi_c + \frac{\lambda\phi_c}{2\pi\varepsilon E_o}$$

$$y^* = 2a\sin\phi_c + \frac{\lambda(\phi_c - \pi)}{2\pi\varepsilon E_o}$$

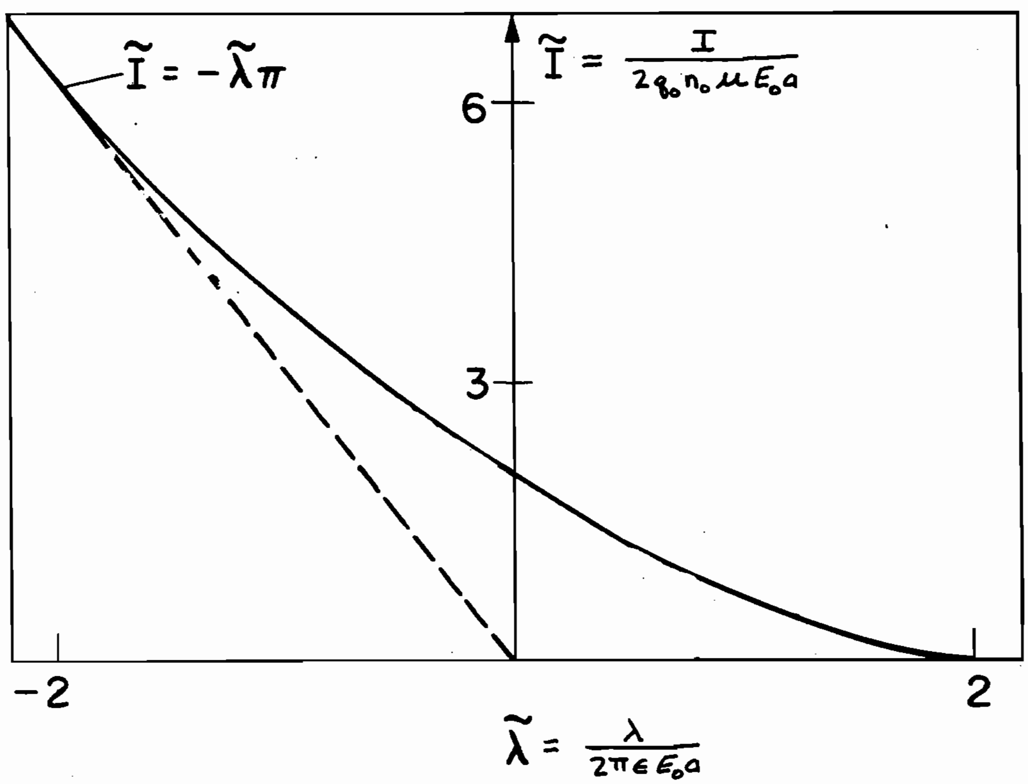
$$I = q_o n_o \mu E_o(2y^*)$$

$$= 2q_o n_o \mu E_o a[2\sin\phi_c + \frac{\lambda(\phi_c - \pi)}{2\pi\varepsilon E_o a}] \qquad \text{(agrees with d)}$$



x
φ
$\tilde{\lambda} = 2$
$\tilde{\lambda} = 1$
$\tilde{\lambda} = 0$
ϕ_c
$\tilde{\lambda} = -1$
$\tilde{\lambda} = -2$

(a)



$\tilde{I} = -\tilde{\lambda}\pi$
$\tilde{I} = \frac{I}{2 q_0 n_0 \mu E_0 a}$
6
3
−2
2
$\tilde{\lambda} = \frac{\lambda}{2\pi\epsilon E_0 a}$

(d), (e)

(d)

14. a) $\sigma_f(t) = \sigma_f(t=0)e^{-t/\tau}; \quad \tau = \dfrac{\varepsilon_1 + \varepsilon_2}{\sigma_1 + \sigma_2}$

$$= \frac{2(\sigma_2\varepsilon_1 - \sigma_1\varepsilon_2)}{\sigma_1 + \sigma_2} E_o \cos\phi e^{-t/\tau}$$

b) $V = \mathrm{Re}\hat{V}(r,\phi)e^{j\omega t}$

$$\hat{V}(r,\phi) = \begin{cases} \hat{A}r\cos\phi & 0 \le r \le a \\ [\hat{B}r + \dfrac{\hat{C}}{r}]\cos\phi & r \ge a \end{cases}$$

$$\hat{\bar{E}} = -\nabla\hat{V} = \begin{cases} -\hat{A}[\cos\phi\bar{i}_r - \sin\phi\bar{i}_\phi] & 0 \le r < a \\ -[\hat{B} - \dfrac{\hat{C}}{r^2}]\cos\phi\bar{i}_r + [\hat{B} + \dfrac{\hat{C}}{r^2}]\sin\phi\bar{i}_\phi & r > a \end{cases}$$

$$\hat{\bar{E}}(r\to\infty) = E_o(\bar{i}_r\cos\phi - \bar{i}_\phi\sin\phi) \to \hat{B} = -E_o$$

$$\hat{V}(r=a_+) = \hat{V}(r=a_-) \to \hat{A}a = \hat{B}a + \frac{\hat{C}}{a}$$

$$\hat{J}_r(r=a_+) - \hat{J}_r(r=a_-) + j\omega\hat{\sigma}_f = 0 \to [\sigma_1 + j\omega\varepsilon_1]\hat{E}_r(r=a_+) = [\sigma_2 + j\omega\varepsilon_2]\hat{E}_r(r=a_-)$$

$$\hat{C} = a^2E_o\frac{[\sigma_2 - \sigma_1 + j\omega(\varepsilon_2 - \varepsilon_1)]}{\sigma_2 + \sigma_1 + j\omega(\varepsilon_2 + \varepsilon_1)}, \quad \hat{B} = -E_o, \quad \hat{A} = \frac{2E_o(\sigma_1 + j\omega\varepsilon_1)}{\sigma_1 + \sigma_2 + j\omega(\varepsilon_1 + \varepsilon_2)}$$

$$\hat{\sigma}_f = \varepsilon_1\hat{E}_r(r=a_+) - \varepsilon_2\hat{E}_r(r=a_-)$$

$$= [-\varepsilon_1(\hat{B} - \frac{\hat{C}}{a^2}) + \varepsilon_2\hat{A}]\cos\phi$$

$$= \frac{2E_o[\varepsilon_1\sigma_2 - \varepsilon_2\sigma_1]\cos\phi}{\sigma_1 + \sigma_2 + j\omega(\varepsilon_1 + \varepsilon_2)}$$

15. a)

$$V(r,z) = \begin{cases} A_1z + B_1z\ln r + C_1\ln r + D_1 & a \le r \le b \\ A_2z + B_2z\ln r + C_2\ln r + D_2 & b \le r \le c \end{cases}$$

Boundary conditions:

$$\begin{aligned} V(r,z=0) &= 0 \quad & 0 \le r \le c \\ V(r=a,z) &= 0 & 0 \le z \le \ell \\ V(r,z=\ell) &= V_o & b \le r \le c \\ J_r(r=b,z) &= 0 & 0 \le z \le \ell \end{aligned} \quad \rightarrow \quad V(r,z) = \begin{cases} \dfrac{V_o z}{\ell \, \ln \frac{b}{a}} \ln \dfrac{r}{a} & a \le r \le b \\[2ex] \dfrac{V_o z}{\ell} & b \le r \le c \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} \dfrac{-V_o}{\ell \, \ln \frac{b}{a}} [\ln \dfrac{r}{a} \overline{i}_z + \dfrac{z}{r} \overline{i}_r] & a < r < b \\[2ex] \dfrac{-V_o}{\ell} \overline{i}_z & b \le r \le c \end{cases}$$

b) $\sigma_f(r=a) = \varepsilon_o E_r(r=a) = \dfrac{-\varepsilon_o V_o}{\ell \, \ln \frac{b}{a}} \dfrac{z}{a}$

$$\sigma_f(r=b) = \varepsilon E_r(r=b_+) - \varepsilon_o E_r(r=b_-) = \frac{\varepsilon_o V_o}{\ell \, \ln \frac{b}{a}} \frac{z}{b}$$

$$\sigma_f(z=0) = \begin{cases} \varepsilon_o E_z(z=0) = \dfrac{-\varepsilon_o V_o}{\ell \, \ln \frac{b}{a}} \ln \dfrac{r}{a} & a < r < b \\[2ex] \varepsilon E_z(z=0) = \dfrac{-\varepsilon V_o}{\ell} & b \le r \le c \end{cases}$$

c) $\dfrac{dr}{dz} = \dfrac{E_r}{E_z} = \dfrac{z}{r \ln \frac{r}{a}} \rightarrow z^2 = r^2 [\ln \dfrac{r}{a} - \dfrac{1}{2}] + \text{constant}$

16. a) $V(r,z) = \begin{cases} A_1 z + B_1 z \ln r + C_1 \ln r + D_1 & 0 \le r \le a \\ A_2 z + B_2 z \ln r + C_2 \ln r + D_2 & a \le r \le b \end{cases}$

<u>Boundary conditions</u>:

$V(z=0) = 0 \rightarrow C_1 = D_1 = 0, \quad C_2 = D_2 = 0$

$V(r=b) = 0 \rightarrow A_2 + B_2 \ln b = 0$

$V(z=\ell, r<a) = V_o \rightarrow A_1 \ell = V_o, \quad B_1 = 0$

$V(r=a_+) = V(r=a_-) \rightarrow A_2 + B_2 \ln a = A_1$

$$A_1 = \frac{V_o}{\ell}, \; B_1 = C_1 = D_1 = C_2 = D_2 = 0, \; B_2 = \frac{V_o}{\ell \, \ell n \frac{a}{b}}, \quad A_2 = -\frac{V_o \ell n \, b}{\ell \, \ell n \frac{a}{b}}$$

$$V(r,z) = \begin{cases} \dfrac{V_o z}{\ell} \\ \dfrac{V_o z \, \ell n \frac{b}{r}}{\ell \, \ell n \frac{b}{a}} \end{cases} ; \quad \overline{E} = -\nabla V = \begin{cases} -\dfrac{V_o}{\ell} \overline{i}_z & 0 \leq r \leq a \\ -\dfrac{V_o}{\ell \, \ell n \frac{b}{a}} [\ell n \frac{b}{r} \overline{i}_z - \frac{z}{r} \overline{i}_r] & a < r < b \end{cases}$$

b) $\sigma_f(r=a) = \varepsilon_o E_r(r=a_+) - \varepsilon E_r(r=a_-)$

$$= \frac{\varepsilon_o V_o}{\ell \, \ell n \frac{b}{a}} \frac{z}{a}$$

$$q_T(r=a) = 2\pi a \int_0^{\ell} \sigma_f dz$$

$$= \frac{\pi \varepsilon_o V_o \ell}{\ell n \frac{b}{a}}$$

c) $\dfrac{dr}{dz} = \dfrac{E_r}{E_z} = \dfrac{-z}{r \ell n \frac{b}{r}} \rightarrow z^2 = r^2[\ell n \frac{r}{b} - \frac{1}{2}] + \text{constant}$

Section 4.4

17. a) $\overline{E} = E_o\{(1 + \frac{2R^3}{r^3})\cos\theta \overline{i}_r - (1 - \frac{R^3}{r^3})\sin\theta \overline{i}_\theta\}$

$$\sigma_f = \varepsilon_o E_r(r=R) = 3\varepsilon_o E_o \cos\theta$$

$$q_t = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sigma_f R^2 \sin\theta d\theta d\phi$$

$$= 3\varepsilon_o E_o R^2 2\pi \int_{\theta=0}^{\pi/2} \sin\theta\cos\theta d\theta$$

$$= -3E_o R^2 2\pi\varepsilon_o \left. \frac{\cos 2\theta}{4} \right|_{\theta=0}^{\pi/2}$$

$$= 3\pi\varepsilon_o E_o R^2$$

If the hemisphere were not there $q_t = \pi R^2 \varepsilon_o E_o$ so that the hemisphere has three times more charge.

b) $$F_z = \int_{\theta=0}^{\pi/2} \frac{1}{2}\varepsilon E_r^2(r=R)\cos\theta 2\pi R^2 \sin\theta d\theta$$

$$= 9\pi\varepsilon R^2 E_o^2 \int_{\theta=0}^{\pi/2} \cos^3\theta \sin\theta d\theta$$

$$= 9\pi\varepsilon R^2 E_o^2 \left.\frac{(-\cos^4\theta)}{4}\right|_0^{\pi/2}$$

$$= \frac{9}{4}\pi\varepsilon R^2 E_o^2 \geq \frac{2}{3}\pi R^3 \rho_m g \rightarrow E_o \geq \sqrt{\frac{8\rho_m g R}{27\varepsilon}}$$

18. a) $$V(r,\theta) = \begin{cases} A(t) r\cos\theta & r \leq R \\ [B(t) r + \frac{C(t)}{r^2}]\cos\theta & r \geq R \end{cases}$$

$$\bar{E} = -\nabla V = \begin{cases} -A(t)[\bar{i}_r \cos\theta - \bar{i}_\theta \sin\theta] & r < R \\ -(B(t) - \frac{2C(t)}{r^3})\cos\theta \bar{i}_r + (B(t) + \frac{C(t)}{r^3})\sin\theta \bar{i}_\theta & r > R \end{cases}$$

Boundary conditions:

$$\bar{E}(r\to\infty) = E_o(\bar{i}_r\cos\theta - \bar{i}_\theta \sin\theta) \rightarrow B(t) = -E_o$$

$$V(r=R_+) = V(r=R_-) \rightarrow A(t)R = BR + \frac{C(t)}{R^2}$$

$$\sigma_1 E_r(r=R_+) - \sigma_2 E_r(r=R_-) + \frac{\partial}{\partial t}[\varepsilon_1 E_r(r=R_+) - \varepsilon_2 E_r(r=R_-)] = 0$$

Initial conditions:

$$\sigma_f(t=0) = \varepsilon_1 E_r(r=R_+) - \varepsilon_2 E_r(r=R_-) = 0$$

$$A(t=0) = -\frac{3\varepsilon_1 E_o}{2\varepsilon_1 + \varepsilon_2},\ B = -E_o,\ C(t=0) = \frac{(\varepsilon_2 - \varepsilon_1)R^3}{2\varepsilon_1 + \varepsilon_2} E_o$$

b) $$\frac{\partial C}{\partial t} + \frac{C}{\tau} = \frac{(\sigma_2 - \sigma_1)E_o R^3}{2\varepsilon_1 + \varepsilon_2}\ ;\ \tau = \frac{2\varepsilon_1 + \varepsilon_2}{2\sigma_1 + \sigma_2}$$

$$C(t) = E_o R^3 \left\{ \frac{\sigma_2 - \sigma_1}{2\sigma_1 + \sigma_2} (1 - e^{-t/\tau}) + \frac{(\varepsilon_2 - \varepsilon_1)}{2\varepsilon_1 + \varepsilon_2} e^{-t/\tau} \right\}$$

$$A(t) = -E_o + \frac{C(t)}{R^3}$$

c) $$\sigma_f = \varepsilon_1 E_r(r=R_+) - \varepsilon_2 E_r(r=R_-)$$

$$= \left\{\varepsilon_1\left[-B + \frac{2C}{R^3}\right] + \varepsilon_2 A\right\} \cos\theta$$

$$= \left\{(\varepsilon_1 - \varepsilon_2)E_o + (2\varepsilon_1 + \varepsilon_2) \frac{C}{R^3}\right\} \cos\theta$$

$$= \frac{3(\varepsilon_1\sigma_2 - \varepsilon_2\sigma_1)}{2\sigma_1 + \sigma_2} E_o \cos\theta(1 - e^{-t/\tau})$$

d) Charge conservation boundary condition at $r = R$ becomes

$$[\sigma_1 + j\omega\varepsilon_1]\hat{E}_r(r=R_+) = [\sigma_2 + j\omega\varepsilon_2]\hat{E}_r(r=R_-); \; \overline{E} = \mathrm{Re}\,\hat{\overline{E}}\, e^{j\omega t}$$

The solution is the same as letting

$$\varepsilon_1 \rightarrow \sigma_1 + j\omega\varepsilon_1, \quad \varepsilon_2 \rightarrow \sigma_2 + j\omega\varepsilon_2$$

in the initial condition solutions of (a)

$$\hat{A} = \frac{-3[\sigma_1 + j\omega\varepsilon_1]}{2\sigma_1 + \sigma_2 + j\omega(2\varepsilon_1 + \varepsilon_2)}, \quad \hat{B} = -E_o, \quad \hat{C} = \frac{[(\sigma_2 - \sigma_1) + j\omega(\varepsilon_2 - \varepsilon_1)]R^3 E_o}{2\sigma_1 + \sigma_2 + j\omega(2\varepsilon_1 + \varepsilon_2)}$$

19.

$$V(r,\theta) = \begin{cases} Ar^2(3\cos^2\theta - 1) & r \le R \\ \frac{B}{r^3}(3\cos^2\theta - 1) & r \ge R \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} -2Ar[(3\cos^2\theta - 1)\overline{i}_r - 3\cos\theta\sin\theta\,\overline{i}_\theta] & r < R \\ \frac{3B}{r^4}[(3\cos^2\theta - 1)\overline{i}_r + 2\sin\theta\cos\theta\,\overline{i}_\theta] & r > R \end{cases}$$

Boundary conditions:

$$V(r=R_+) = V(r=R_-) \rightarrow AR^2 = \frac{B}{R^3}$$

$$\varepsilon_1 E_r(r=R_+) - \varepsilon_2 E_r(r=R_-) = \sigma_o(3\cos^2\theta - 1) \rightarrow \frac{3B}{R^4}\varepsilon_1 + 2AR\varepsilon_2 = \sigma_o$$

$$B = \frac{\sigma_o R^4}{[3\varepsilon_1 + 2\varepsilon_2]}, \qquad A = \frac{\sigma_o}{R[3\varepsilon_1 + 2\varepsilon_2]}$$

20.

$$V(r,\theta) = \begin{cases} Ar\cos\theta & r \le R \\ (Br + \frac{C}{r^2})\cos\theta & r \ge R \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} -A(\overline{i}_r\cos\theta - \overline{i}_\theta\sin\theta) & r < R \\ -(B - \frac{2C}{r^3})\cos\theta\overline{i}_r + (B + \frac{C}{r^3})\sin\theta\overline{i}_\theta & r > R \end{cases}$$

Boundary conditions:

$$V(r=R_+) = V(r=R_-) \rightarrow AR = BR + \frac{C}{R^2}$$

$$\overline{E}(r\rightarrow\infty) = E_o(\overline{i}_r\cos\theta - \overline{i}_\theta\sin\theta) \rightarrow B = -E_o$$

$$\varepsilon_o E_r(r=R_+) + P_1\cos\theta = \varepsilon_o E_r(r=R_-) + P_2\cos\theta$$

$$A = -E_o + \frac{(P_2 - P_1)}{3\varepsilon_o}, \quad B = -E_o, \quad C = \frac{(P_2 - P_1)R^3}{3\varepsilon_o}$$

21.

$$V(r,\theta) = \begin{cases} (Ar + \frac{B}{r^2})\cos\theta & r \le R \\ \frac{C}{r^2}\cos\theta & r \ge R \end{cases}$$

$$\overline{E} = -\nabla V = \begin{cases} -(A - \frac{2B}{r^3})\cos\theta\overline{i}_r + (A + \frac{B}{r^3})\sin\theta\overline{i}_\theta & r < R \\ \frac{C}{r^3}[2\cos\theta\overline{i}_r + \sin\theta\overline{i}_\theta] & r > R \end{cases}$$

Boundary conditions:

$$\lim_{r\to 0} V(r,\theta) = \frac{p}{4\pi\varepsilon_2}\frac{\cos\theta}{r^2} \rightarrow B = \frac{p}{4\pi\varepsilon_2}$$

$$V(r=R_+) = V(r=R_-) \rightarrow AR + \frac{B}{R^2} = \frac{C}{R^2}$$

$$\varepsilon_1 E_r(r=R_+) = \varepsilon_2 E_r(r=R_-) \rightarrow -\varepsilon_2\left(A - \frac{2B}{R^3}\right) = \frac{2C}{R^3}\varepsilon_1$$

$$A = \frac{2p(\varepsilon_2 - \varepsilon_1)}{4\pi\varepsilon_2 R^3(\varepsilon_2 + 2\varepsilon_1)},\quad B = \frac{p}{4\pi\varepsilon_2},\quad C = \frac{3\varepsilon_2}{\varepsilon_2 + 2\varepsilon_1}\frac{p}{4\pi\varepsilon_2} = \frac{3p}{4\pi(\varepsilon_2 + 2\varepsilon_1)}$$

Section 4.5

22.
```
      DIMENSION V(4,4,20), P(4,4,20)
      READ (5,30) V1, V2, V3, V4
      WRITE (6,9)
      WRITE (6,8) V1, V2, V3, V4
      WRITE (6,9)
      DO 10 K=1,10
      DO 10 J=1,4
      DO 10 I=1,4
      V(I,J,K)=0.  ;  P(I,J,K)=0.
      V(I,1,K)=V1  ;  V(4,J,K)=V2  ;  V(I,4,K)=V3  ;  V(1,J,K)=V4
   10 CONTINUE
      DO  3 K=1,10
      DO 3 J=2,3
      DO 3 I=2,3
      V(I,J,K+1)=0.25*(V(I+1,J,K)+V(I-1,J,K)+V(I,J+1,K)+V(I,J-1,K))-2.
      P(I,J,K)=V(I,J,K)   ;   V(I,J,K)=V(I,J,K+1)
    3 CONTINUE
      WRITE (6,7) (K,K=1,10)
      WRITE (6,9)
      WRITE (6,6) (P(2,2,K),K=1,10)
      WRITE (6,4)
      WRITE (6,6) (P(3,2,K),K=1,10)
      WRITE (6,4)
      WRITE (6,6) (P(2,3,K),K=1,10)
      WRITE (6,4)
      WRITE (6,6) (P(3,3,K),K=1,10)
      WRITE (6,9)  ;  WRITE (6,9)  ;  WRITE (6,9)
    4 FORMAT (' ')
    6 FORMAT ('+',10(F11.4,2X))
    7 FORMAT ('+',6X,9(I2,11X),I2)
    8 FORMAT ('0V1 = ',F7.2,4X,'V2 = ',F7.2,4X,'V3 = ',F7.2,4X,'V4 = ',
     &F7.2)
    9 FORMAT ('0')
```

```
30 FORMAT (4F10.2)
   STOP
   END
```

```
V1  =  0.00    V2  =  0.00    V3  =  0.00    V4  =  0.00
```

K	1	2	3	4	5
V(2,2)	0.0000	-2.0000	-3.2500	-3.8125	-3.9531
V(3,2)	0.0000	-2.5000	-3.6250	-3.9063	-3.9766
V(2,3)	0.0000	-2.5000	-3.6250	-3.9063	-3.9766
V(3,3)	0.0000	-3.2500	-3.8125	-3.9531	-3.9883

	6	7	8	9	10
	-3.9883	-3.9971	-3.9993	-3.9998	-4.0000
	-3.9941	-3.9985	-3.9996	-3.9999	-4.0000
	-3.9941	-3.9985	-3.9996	-3.9999	-4.0000
	-3.9971	-3.9993	-3.9998	-4.0000	-4.0000

23.
```
     DIMENSION V(4,4,20), P(4,4,20)
   1 DO 5 M=1,2
     READ (5,30) V1, V2, V3, V4
     WRITE (6,9)
     WRITE (6,8) V1, V2, V3, V4
     WRITE (6,9)
     DO 10 K=1,10
     DO 10 J=1,4
     DO 10 I=1,4
     V(I,J,K)=0.   ;   P(I,J,K)=0.
     V(I,1,K)=V1  ;  V(4,J,K)=V2  ;  V(I,4,K)=V3  ;  V(1,J,K)=V4
  10 CONTINUE
     DO 3 K=1,10
     DO 3 J=2,3
     DO 3 I=2,3
     V(I,J,K+1)=0.25*(V(I+1,J,K)+V(I-1,J,K)+V(I,J+1,K)+V(I,J-1,K))
     P(I,J,K)=V(I,J,K)   ;   V(I,J,K)=V(I,J,K+1)
   3 CONTINUE
     WRITE (6,7) (K,K=1,10)
     WRITE (6,9)
     WRITE (6,6) (P(2,2,K),K=1,10)
     WRITE (6,4)
     WRITE (6,6) (P(3,2,K),K=1,10)
     WRITE (6,4)
     WRITE (6,6) (P(2,3,K),K=1,10)
     WRITE (6,4)
     WRITE (6,6) (P(3,3,K),K=1,10)
   5 CONTINUE
     WRITE (6,9)  ;  WRITE (6,9)  ; WRITE (6,9)
   4 FORMAT (' ')
   6 FORMAT ('+',10(F11.4,2X))
   7 FORMAT ('+',6X,9(I2,11X),I2)
```

```
 8 FORMAT ('0V1 = ',F7.2,4X,'V2 = ',F7.2,4X,'V3 = ',F7.2,4X,'V4 = ',
  &F7.2)
 9 FORMAT ('0')
30 FORMAT (4F10.2)
   STOP
   END
```

```
V1  =  1.00    V2  =  -2.00    V3  =  3.00    V4  =  -4.00

  K          1         2         3         4         5

V(2,2)   0.0000   -0.7500   -0.9688   -0.9922   -0.9980
V(3,2)   0.0000   -0.4375   -0.4844   -0.4961   -0.4990
V(2,3)   0.0000   -0.4375   -0.4844   -0.4961   -0.4990
V(3,3)   0.0000    0.0313    0.0078    0.0020    0.0005

             6         7         8         9        10

        -0.9995   -0.9999   -1.0000   -1.0000   -1.0000
        -0.4998   -0.4999   -0.5000   -0.5000   -0.5000
        -0.4998   -0.4999   -0.5000   -0.5000   -0.5000
         0.0001    0.0000    0.0000    0.0000    0.0000


V1  =  1.00    V2  =  -2.00    V3  =  -3.00    V4  =  4.00

  K          1         2         3         4         5

V(2,2)   0.0000    1.2500    1.4063    1.2891    1.2598
V(3,2)   0.0000    0.0625   -0.1719   -0.2305   -0.2451
V(2,3)   0.0000    0.5625    0.3281    0.2695    0.2549
V(3,3)   0.0000   -1.0938   -1.2109   -1.2402   -1.2476

             6         7         8         9        10

         1.2524    1.2506    1.2502    1.2500    1.2500
        -0.2488   -0.2497   -0.2499   -0.2500   -0.2500
         0.2512    0.2503    0.2501    0.2500    0.2500
        -1.2494   -1.2498   -1.2500   -1.2500   -1.2500
```

CHAPTER 5
THE MAGNETIC FIELD

Section 5.1

1. a) $v_x = v_{xo}\cos\omega_o t + v_{yo}\sin\omega_o t$; $\omega_o = qB_o/m$

$$v_y = v_{yo}\cos\omega_o t - v_{xo}\sin\omega_o t$$

$$v_z = v_{zo}$$

$$x = \int v_x dt = x_o + \frac{v_{yo}}{\omega_o} + \frac{1}{\omega_o}[-v_{yo}\cos\omega_o t + v_{xo}\sin\omega_o t]$$

$$y = \int v_y dt = y_o - \frac{v_{xo}}{\omega_o} + \frac{1}{\omega_o}[v_{yo}\sin\omega_o t + v_{xo}\cos\omega_o t]$$

$$z = \int v_z dt = v_{zo}t + z_o$$

b) $$[x - (x_o + \frac{v_{yo}}{\omega_o})]^2 + [y - (y_o - \frac{v_{xo}}{\omega_o})]^2 = \frac{v_{xo}^2 + v_{yo}^2}{\omega_o^2}$$

Equation of circle with radius $r = [(v_{xo}^2 + v_{yo}^2)/\omega_o^2]^{1/2}$ with center at $(x_o + \frac{v_{yo}}{\omega_o}, y_o - \frac{v_{xo}}{\omega_o})$.

c) $$\frac{1}{2}m|\bar{v}|^2 = \frac{1}{2}m[v_{xo}^2 + v_{yo}^2 + v_{zo}^2]$$

2. a) $$m\frac{dv_x}{dt} = -e(-\frac{V_o}{s} + v_y B_o)$$

$$m\frac{dv_y}{dt} = ev_x B_o \rightarrow v_x = \frac{m}{eB_o}\frac{dv_y}{dt}$$

$$\frac{d^2v_y}{dt^2} + \omega_o^2 v_y = \frac{\omega_o^2 V_o}{B_o s} \; ; \quad \omega_o = \frac{eB_o}{m}$$

$$v_y = A_1 \sin\omega_o t + A_2 \cos\omega_o t + \frac{V_o}{B_o s}$$

$$v_y(t=0) = 0 = A_2 + \frac{V_o}{B_o s}$$

$$v_x(t=0) = 0 \rightarrow \left.\frac{dv_y}{dt}\right|_{t=0} = 0 \rightarrow A_1 = 0$$

$$v_y = \frac{V_o}{B_o s}(1 - \cos\omega_o t), \quad v_x = \frac{1}{\omega_o}\frac{dv_y}{dt} = \frac{V_o}{B_o s}\sin\omega_o t$$

$$x = \int v_x dt = \frac{V_o}{B_o s\omega_o}(1 - \cos\omega_o t), \quad y = \int v_y dt = \frac{V_o}{B_o s}\left(t - \frac{\sin\omega_o t}{\omega_o}\right)$$

b) $$x_{max} = \frac{2V_o}{B_o s\omega_o} < s \rightarrow B_o^2 > \frac{2mV_o}{es^2} \quad \text{for cut-off}$$

c) $$\bar{i}_r = \cos\phi\,\bar{i}_x + \sin\phi\,\bar{i}_y, \quad \bar{i}_\phi = -\sin\phi\,\bar{i}_x + \cos\phi\,\bar{i}_y$$

$$\frac{d\bar{i}_r}{dt} = [-\sin\phi\,\bar{i}_x + \cos\phi\,\bar{i}_y]\frac{d\phi}{dt} = \bar{i}_\phi \frac{v_\phi}{r}$$

$$\frac{d\bar{i}_\phi}{dt} = [-\cos\phi\,\bar{i}_x - \sin\phi\,\bar{i}_y]\frac{d\phi}{dt} = -\bar{i}_r \frac{v_\phi}{r}$$

$$\bar{a} = \frac{d\bar{v}}{dt} = \bar{i}_r \frac{dv_r}{dt} + v_r \frac{d\bar{i}_r}{dt} + \bar{i}_\phi \frac{dv_\phi}{dt} + v_\phi \frac{d\bar{i}_\phi}{dt}$$

$$= \bar{i}_r \left[\frac{dv_r}{dt} - \frac{v_\phi^2}{r}\right] + \bar{i}_\phi \left[\frac{dv_\phi}{dt} + \frac{v_\phi v_r}{r}\right]$$

d) $$m\frac{d\bar{v}}{dt} = -e[\bar{E} + \bar{v} \times \bar{B}]; \quad \bar{E} = \frac{-V_o}{r\ln\frac{b}{a}}\bar{i}_r$$

$$\left[\frac{dv_r}{dt} - \frac{v_\phi^2}{r}\right] = \frac{-eE_r}{m} - \frac{ev_\phi B_o}{m} = \frac{eV_o}{mr\ln\frac{b}{a}} - \omega_o v_\phi; \quad \omega_o = \frac{eB_o}{m}$$

$$\left[\frac{dv_\phi}{dt} + \frac{v_\phi v_r}{r}\right] = \frac{ev_r B_o}{m} = \omega_o v_r$$

$$\frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr} = \frac{d}{dr}\left(\frac{1}{2}v_r^2\right)$$

$$\frac{dv_\phi}{dt} = \frac{dv_\phi}{dr}\frac{dr}{dt} = v_r\frac{dv_\phi}{dr}$$

$$v_r\left[\frac{dv_\phi}{dr} + \frac{v_\phi}{r}\right] = \omega_o v_r \rightarrow \frac{1}{r}\frac{d}{dr}(rv_\phi) = \omega_o \rightarrow rv_\phi = \frac{\omega_o r^2}{2} + \text{constant}$$

$$v_\phi(r=a) = 0 \rightarrow v_\phi = \frac{\omega_o}{2}\left(r - \frac{a^2}{r}\right)$$

$$\frac{d}{dr}\left(\frac{1}{2}v_r^2\right) - \frac{\omega_o^2}{4r}\left(r - \frac{a^2}{r}\right)^2 = \frac{eV_o}{m\ln\frac{b}{a}\,r} - \frac{\omega_o^2}{2}\left(r - \frac{a^2}{r}\right)$$

$$\frac{d}{dr}\left(\frac{1}{2}v_r^2\right) = \frac{eV_o}{m\ln\frac{b}{a}\,r} - \frac{\omega_o^2}{4}\left(r - \frac{a^2}{r}\right)\left(1 + \frac{a^2}{r^2}\right) = \frac{eV_o}{m\ln\frac{b}{a}\,r} - \frac{\omega_o^2}{4}\,r\left(1 - \frac{a^4}{r^4}\right)$$

$$\frac{1}{2}v_r^2 = \left[\frac{eV_o}{m\ln\frac{b}{a}}\right]\ln\frac{r}{a} - \frac{\omega_o^2}{4}\left[\frac{r^2}{2} + \frac{a^4}{2r^2} - a^2\right]$$

$$= \frac{eV_o}{m\ln\frac{b}{a}}\ln\frac{r}{a} - \frac{\omega_o^2}{8r^2}\left[r^2 - a^2\right]^2$$

e) Cut-off $\rightarrow v_r(r=b) < 0 \rightarrow B_o^2 > \dfrac{8b^2 mV_o}{e(b^2 - a^2)^2}$

Check: Cylindrical geometry approaches planar geometry of (a) if $b \approx a$ so that $b = a + s$ where $s << a$.

$$B_o^2 > \frac{8a^2 mV_o}{e4a^2 s^2}$$

$$> \frac{2mV_o}{es^2} \quad \text{(agrees with (b))}$$

3. $m\dfrac{d\overline{v}}{dt} = q\overline{v} \times \overline{B} - mg\overline{i}_y \rightarrow -qv_oB_o - mg = 0 \rightarrow B_o = -\dfrac{mg}{qv_o}$

4. a) $\dfrac{1}{2}mv_x^2 - eV_1 = \dfrac{1}{2}mv_o^2 \rightarrow v_x^2 = v_o^2 + \dfrac{2e}{m}V_1$

b) $E_y - v_x B_o = 0 \rightarrow v_x = \frac{E_y}{B_o} = \sqrt{v_o^2 + \frac{2e}{m} V_1}$; $E_y = \frac{V_2}{s}$

$$v_o^2 = \left(\frac{E_y}{B_o}\right)^2 - \frac{2e}{m} V_1$$

c) and d) $R = \frac{mv_x}{eB_o} = \frac{mE_y}{eB_o^2} \rightarrow \frac{e}{m} = \frac{E_y}{RB_o^2}$

5. a) $m \frac{dv_x}{dt} = 0, \; m \frac{dv_y}{dt} = qv_z B_x, \; m \frac{dv_z}{dt} = -qv_y B_x$

$$v_x = v_{xo}, \; v_y = v_{yo}\cos\omega_o t, \; v_z = -v_{yo}\sin\omega_o t$$

$$x = v_{xo}t, \; y = \frac{v_{yo}}{\omega_o} \sin\omega_o t, \; z = \frac{v_{yo}}{\omega_o}(\cos\omega_o t - 1)$$

$$y = 0 \rightarrow \omega_o t = n\pi \rightarrow \frac{\omega_o x}{v_{xo}} = n\pi \rightarrow x = \frac{n\pi v_{xo}}{\omega_o}, \; z = \frac{-2v_{yo}}{\omega_o} \; \text{(n odd)}$$

$$= 0 \qquad \text{(n even)}$$

b) $m \frac{dv_x}{dt} = -qv_z B_o, \; m \frac{dv_y}{dt} = 0, \; m \frac{dv_z}{dt} = qv_x B_o$

$$v_x = v_{xo}\cos\omega_o t, \; v_y = v_{yo}, \; v_z = v_{xo}\sin\omega_o t$$

$$x = \frac{v_{xo}}{\omega_o} \sin\omega_o t, \; y = v_{yo}t, \; z = \frac{v_{xo}}{\omega_o}(-\cos\omega_o t + 1)$$

$y = 0$ (never for $t > 0$).

c) $m \frac{dv_x}{dt} = qv_y B_z, \; m \frac{dv_y}{dt} = -qv_x B_z, \; m \frac{dv_z}{dt} = 0$

$$v_x = v_{yo}\sin\omega_o t + v_{xo}\cos\omega_o t, \; v_y = v_{yo}\cos\omega_o t - v_{xo}\sin\omega_o t, \; v_z = 0$$

$$x = \frac{1}{\omega_o}[-v_{yo}\cos\omega_o t + v_{xo}\sin\omega_o t + v_{yo}], \; y = \frac{1}{\omega_o}[v_{yo}\sin\omega_o t + v_{xo}\cos\omega_o t - v_{xo}], \; z = 0$$

$$y = 0 \rightarrow v_{yo}\sin\omega_o t + v_{xo}\cos\omega_o t = v_{xo}$$

$$v_{xo}\sqrt{1 - \sin^2\omega_o t} = v_{xo} - v_{yo}\sin\omega_o t$$

$$v_{xo}^2(1 - \sin^2\omega_o t) = v_{xo}^2 + v_{yo}^2\sin^2\omega_o t - 2v_{xo}v_{yo}\sin\omega_o t$$

$$\sin^2\omega_o t(v_{xo}^2 + v_{yo}^2) - 2v_{xo}v_{yo}\sin\omega_o t = 0$$

$\sin\omega_o t = 0$ (one solution) $\rightarrow \omega_o t = 2n\pi \quad n = 1,2,...$

$$\sin\omega_o t = \frac{2v_{xo}v_{yo}}{v_{xo}^2 + v_{yo}^2}, \quad \cos\omega_o t = \frac{(v_{xo}^2 - v_{yo}^2)}{v_{xo}^2 + v_{yo}^2}$$

$\sin\omega_o t = 0$, $\cos\omega_o t = 1$ solution $\rightarrow x = 0,\ y = 0,\ z = 0$

Other solution for $y = 0$

$$x = \frac{2}{\omega_o}\frac{v_{xo}^2 v_{yo}}{v_{xo}^2 + v_{yo}^2}$$

6. a) $$m\frac{dv_x}{dt} = -ev_yB_o - kx$$

$$m\frac{dv_y}{dt} = ev_xB_o - ky$$

$$m\frac{dv_z}{dt} = -kz$$

b) $x = \hat{x}e^{st}$, $y = \hat{y}e^{st}$, $z = \hat{z}e^{st}$; $\omega_o = \frac{eB_o}{m}$, $\omega_k = \sqrt{\frac{k}{m}}$

$$\begin{aligned} \frac{d^2x}{dt^2} + \omega_k^2 x &= -\omega_o\frac{dy}{dt} \\ \frac{d^2y}{dt^2} + \omega_k^2 y &= \omega_o\frac{dx}{dt} \\ \frac{d^2z}{dt^2} + \omega_k^2 z &= 0 \end{aligned} \longrightarrow \begin{bmatrix} s^2 + \omega_k^2 & \omega_o s & 0 \\ -\omega_o s & s^2 + \omega_k^2 & 0 \\ 0 & 0 & s^2 + \omega_k^2 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = 0$$

Determinant of coefficients must be zero

$$(s^2 + \omega_k^2)[(s^2 + \omega_k^2)^2 + \omega_o^2 s^2] = 0$$

$$s^2 + \omega_k^2 = 0 \rightarrow s = \pm j\omega_k$$

$$(s^2 + \omega_k^2)^2 = -\omega_o^2 s^2 \rightarrow s^2 + \omega_k^2 = \pm j\omega_o s \rightarrow s = \pm \frac{j\omega_o}{2} \pm \sqrt{(\frac{\omega_o}{2})^2 - \omega_k^2}$$

c) $$\omega_o = \frac{eB_o}{m} \approx \frac{(1.6 \times 10^{-19})(1)}{9.1 \times 10^{-31}} \approx 1.76 \times 10^{11} << 10^{15} \rightarrow \omega_o << \omega_k$$

$$s = \pm j[\omega_k \pm \omega_o]$$

7. $$\beta\bar{v} = q(\bar{E} + \bar{v} \times \bar{B})$$

$$\beta(\bar{v} \times \bar{B}) = q(\bar{E} \times \bar{B}) + q(\bar{v} \times \bar{B}) \times \bar{B}$$

$$= q(\bar{E} \times \bar{B}) + q[-\bar{v}(\bar{B}\cdot\bar{B}) + \bar{B}(\bar{v}\cdot\bar{B})]$$

$$\bar{v}\cdot\bar{B} = \frac{q}{\beta}\,\bar{E}\cdot\bar{B}$$

$$\beta(\bar{v} \times \bar{B}) = q(\bar{E} \times \bar{B}) - q\bar{v}(\bar{B}\cdot\bar{B}) + \frac{q^2}{\beta}\,\bar{B}(\bar{E}\cdot\bar{B}) = \beta[\frac{\beta}{q}\,\bar{v} - \bar{E}]$$

$$\bar{v}[\frac{\beta^2}{q} + q(\bar{B}\cdot\bar{B})] = q[\bar{E} \times \bar{B} + \frac{q\bar{B}}{\beta}\,(\bar{E}\cdot\bar{B}) + \frac{\beta\bar{E}}{q}]$$

$$\bar{v} = \frac{\bar{E} \times \bar{B} + \frac{q\bar{B}}{\beta}\,(\bar{E}\cdot\bar{B}) + \frac{\beta\bar{E}}{q}}{\frac{\beta^2}{q^2} + \bar{B}\cdot\bar{B}}$$

8. a) $$m\frac{d\bar{v}}{dt} = -m\nu\bar{v} + q(\bar{E} + \bar{v} \times \bar{B}) - \frac{1}{n}\,\nabla(nkT)$$

b) $$\bar{v} \approx \frac{q}{m\nu}\,(\bar{E} + \bar{v} \times \bar{B})$$

c) $$\bar{J} = qn\bar{v} = \frac{q^2 n}{m\nu}\,(\bar{E} + \bar{v} \times \bar{B}) = \sigma(\bar{E} + \bar{v} \times \bar{B})$$

d) $J_x = \sigma(-\frac{V_h}{d} + v_y B_o) = \frac{i_h}{A} = \frac{V_h}{R_L A}$

$$V_h[\frac{\sigma}{d} + \frac{1}{R_L A}] = \sigma v_y B_o$$

$$V_h = \frac{\sigma v_y B_o A}{\frac{1}{R_L} + \frac{1}{R_{int}}} \; ; \quad R_{int} = \frac{d}{\sigma A}$$

e) $P = \frac{V_h^2}{R_L} = \frac{(\sigma v_y B_o A) R_{int}^2 R_L}{(R_L + R_{int})^2}$

$$\frac{\partial P}{\partial R_L} = \frac{(\sigma v_y B_o A) R_{int}^2}{(R_L + R_{int})^2} [1 - \frac{2R_L}{R_L + R_{int}}] = 0 \rightarrow R_L = R_{int}$$

<u>Section 5.2</u>

9. a) $\bar{B} = \frac{\mu_o I}{2\pi r} \bar{i}_\phi$

$$\frac{m d\bar{v}}{dt} = q\bar{v} \times \bar{B}$$

$$\frac{dv_r}{dt} - \frac{v_\phi^2}{r} = \frac{d}{dr}(\frac{1}{2} v_r^2) - \frac{v_\phi^2}{r} = -\frac{q}{m} v_z B_\phi$$

$$\frac{dv_\phi}{dt} + \frac{v_\phi v_r}{r} = v_r [\frac{dv_\phi}{dr} + \frac{v_\phi}{r}] = 0 \rightarrow r v_\phi = r_o v_{\phi o}$$

$$\frac{dv_z}{dt} = v_r \frac{dv_z}{dr} = \frac{q}{m} v_r B_\phi \rightarrow \frac{dv_z}{dr} = \frac{q\mu_o I}{m 2\pi r} \rightarrow v_z = \frac{q\mu_o I}{2\pi m} \ln \frac{r}{r_o} + v_{zo}$$

$$\frac{d}{dr}(\frac{1}{2} v_r^2) = \frac{v_\phi^2}{r} - \frac{q}{m} v_z B_\phi = \frac{(r_o v_{\phi o})^2}{r^3} - \frac{q\mu_o I}{m 2\pi r}[\frac{q\mu_o I}{2\pi m} \ln \frac{r}{r_o} + v_{zo}]$$

$$\frac{1}{2} v_r^2 = -\frac{1}{2}(r_o v_{\phi o})^2 (\frac{1}{r^2} - \frac{1}{r_o^2}) - \frac{q\mu_o I v_{zo}}{2\pi m} \ln \frac{r}{r_o} - \frac{1}{2}(\frac{q\mu_o I}{2\pi m})^2 (\ln \frac{r}{r_o})^2 + \frac{1}{2} v_{ro}^2$$

b) $K.E. = \frac{1}{2} m [v_r^2 + v_\phi^2 + v_z^2]$

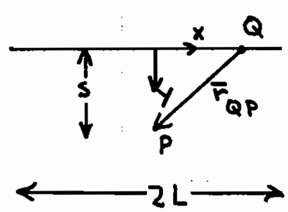
$$= \frac{1}{2} m [v_{ro}^2 + v_{\phi o}^2 + v_{zo}^2]$$

c) With $v_{\phi o} = 0$, for $v_r = 0$

$$\left[\frac{q\mu_o I}{2\pi m} \ln(\frac{r}{r_o})\right]^2 + 2v_{zo}\left[\frac{q\mu_o I}{2\pi m} \ln \frac{r}{ro}\right] - v_{ro}^2 = 0$$

$$\frac{q\mu_o I}{2\pi m} \ln(\frac{r}{r_o}) = -v_{zo} \pm \sqrt{v_{zo}^2 + v_{ro}^2}$$

10. $\overline{B} = \frac{\mu_o}{4\pi} \int_L \frac{\overline{I}d\ell \times \overline{i}_{QP}}{r_{QP}^2}$



$\overline{r}_{QP} = -(x\overline{i}_x - s\overline{i}_y)$

$$\overline{i}_{QP} = -\frac{(x\overline{i}_x - s\overline{i}_y)}{\sqrt{x^2 + s^2}}$$

$$\overline{B} = \frac{\mu_o Is}{4\pi} \overline{i}_z \int_{-L}^{+L} \frac{dx}{(x^2 + s^2)^{3/2}}$$

$$= \frac{\mu_o Is}{4\pi} \overline{i}_z \left. \frac{x}{s^2[x^2 + s^2]^{1/2}} \right|_{-L}^{L}$$

$$= \frac{\mu_o I L}{2\pi s[L^2 + s^2]^{1/2}} \overline{i}_z$$

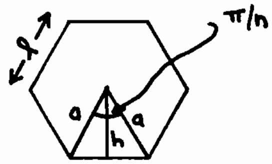
a) Let $L \to a$, $L \to b$ and use superposition

$$\overline{B} = \frac{2\mu_o I}{2\pi} \left\{\frac{a/2}{[(\frac{a}{2})^2+(\frac{b}{2})^2]^{1/2} \frac{b}{2}} + \frac{b/2}{[(\frac{a}{2})^2+(\frac{b}{2})^2]^{1/2} \frac{a}{2}}\right\} \overline{i}_z$$

$$= \frac{2\mu_o I}{\pi[a^2 + b^2]^{1/2}} [\frac{a}{b} + \frac{b}{a}]\overline{i}_z = \frac{2\mu_o I[a^2 + b^2]^{1/2}}{\pi ab} \overline{i}_z$$

b) $\overline{B} = -\frac{\mu_o Ia}{4\pi} \int_0^{2\pi} \frac{\overline{i}_\phi \times \overline{i}_r}{a^2} d\phi$

$$= \frac{\mu_o I}{2a} \overline{i}_z$$

c) 

$$h^2 = a^2 - \frac{\ell^2}{4} = a^2\cos^2\frac{\pi}{n} \rightarrow \left(\frac{\ell}{2}\right)^2 = a^2\left(1 - \cos^2\frac{\pi}{n}\right) = a^2\sin^2\frac{\pi}{n}$$

$$B_z = \frac{n\mu_o I \frac{\ell}{2}}{2\pi h\left[\left(\frac{\ell}{2}\right)^2 + h^2\right]^{1/2}}$$

$$= \frac{n\mu_o I\sin\frac{\pi}{n}}{2\pi a\cos\frac{\pi}{n}} = \frac{n\mu_o I}{2\pi a}\tan\frac{\pi}{n}$$

Check: $\lim n \rightarrow \infty \qquad B_z = \frac{\mu_o I}{2a}$ (agrees with (b))

$\lim n \rightarrow 4 \qquad B_z = \frac{2\mu_o I}{\pi a}$ (agrees with (a) when a = b and a (of (c)) $\rightarrow \sqrt{2}a$ (of (a))

d) No contribution from straight line segments

$$\overline{B} = \frac{\mu_o I}{4\pi}\int_{\phi=0}^{\pi} \frac{a\overline{i}_\phi \times (-\overline{i}_r)d\phi}{a^2}$$

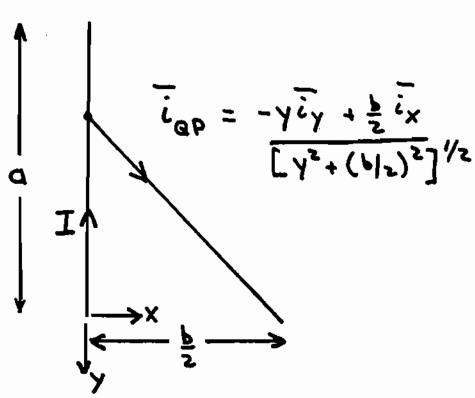
$$= \frac{\mu_o I}{4a}\overline{i}_z$$

e) No contribution from semi-infinite line currents.

From horizontal length

$$B_z = \frac{\mu_o I \frac{b}{2}}{2\pi a\left[\left(\frac{b}{2}\right)^2 + a^2\right]^{1/2}}$$

From vertical length



$$\overline{B} = \frac{\mu_o I}{4\pi}\int_{-a}^{0} \frac{-\overline{i}_y \times \overline{i}_{QP}dy}{\left[y^2 + \left(\frac{b}{2}\right)^2\right]}$$

$$= \frac{\mu_o I\overline{i}_z}{4\pi}\int_{-a}^{0} \frac{\frac{b}{2}\,dy}{\left[y^2 + \left(\frac{b}{2}\right)^2\right]^{3/2}}$$

$$= \frac{\mu_o I}{4\pi}\overline{i}_z\,\frac{b}{2}\,\frac{y}{\left(\frac{b}{2}\right)^2\left[y^2 + \left(\frac{b}{2}\right)^2\right]^{1/2}}\Bigg|_{y=-a}^{0}$$

$$= \frac{\mu_o I a \bar{i}_z}{2\pi b[a^2 + (\frac{b}{2})^2]^{1/2}}$$

$$B_{ztotal} = \frac{\mu_o I}{2\pi\left[a^2 + (\frac{b}{2})^2\right]^{1/2}} \left[\frac{2a}{b} + \frac{b}{2a}\right] = \frac{\mu_o I}{\pi ab} [a^2 + (\frac{b}{2})^2]^{1/2}$$

f) From straight line segments, $B_z = \frac{\mu_o I}{2\pi a}$ (see (e))

From semi-circle, $B_z = \frac{\mu_o I}{4a}$ (see (d))

$$B_{ztotal} = \frac{\mu_o I}{2a} (\frac{1}{2} + \frac{1}{\pi})$$

11. $F_r = \frac{\mu_o I^2}{2\pi(2\ell \sin \frac{\theta}{2})} = T\sin \frac{\theta}{2}, \qquad T\cos \frac{\theta}{2} = mg$

$$T\sin^2 \frac{\theta}{2} = \frac{\mu_o I^2}{4\pi\ell} = \frac{mg\sin^2 \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \frac{mg\sin^2 \frac{\theta}{2}}{\sqrt{1 - \sin^2 \frac{\theta}{2}}}$$

$$\sin^4 \frac{\theta}{2} + (\frac{\mu_o I^2}{4\pi\ell mg})^2 \sin^2 \frac{\theta}{2} - (\frac{\mu_o I^2}{4\pi\ell mg})^2 = 0$$

$$\sin^2 \frac{\theta}{2} = -\frac{1}{2} (\frac{\mu_o I^2}{4\pi\ell mg})^2 \pm \sqrt{[\frac{1}{2} (\frac{\mu_o I^2}{4\pi\ell mg})^2]^2 + (\frac{\mu_o I^2}{4\pi\ell mg})^2}$$

$$= \frac{1}{2} (\frac{\mu_o I^2}{4\pi\ell mg})^2 \left[-1 + \sqrt{1 + 4 \left[\frac{4\pi\ell mg}{\mu_o I^2}\right]^2}\right]$$

12. a) $\overline{dB} = -\frac{\mu_o}{4\pi} K_o \bar{i}_\phi \times \bar{i}_r \sin\theta d\theta d\phi$

$$= -\frac{\mu_o K_o}{4\pi} \sin\theta d\theta d\phi[\bar{i}_x \cos\theta\cos\phi + \cos\theta\sin\phi\bar{i}_y - \sin\theta\bar{i}_z]$$

$$\bar{B} = -\frac{\mu_o K_o}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta d\theta d\phi[\bar{i}_x\cos\theta\cos\phi + \cos\theta\sin\phi\bar{i}_y - \sin\theta\bar{i}_z]$$

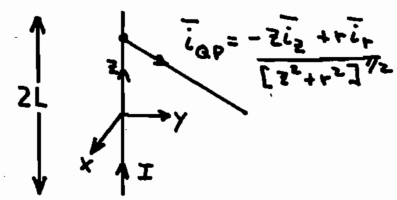
$$= \frac{\mu_o K_o}{2} \int_{\theta=0}^{\pi} \sin^2\theta d\theta \; \bar{i}_z$$

$$= \frac{\mu_o K_o \pi}{4} \bar{i}_z$$

b) $K_o \to J_o dr$

$$B_z = \frac{\mu_o J_o \pi}{4} \int_a^b dr = \frac{\mu_o J_o \pi (b - a)}{4}$$

13. a) $\overline{dB} = \frac{\mu_o I}{4\pi} dz \frac{(\bar{i}_z \times \bar{i}_{QP})}{(z^2 + r^2)}$

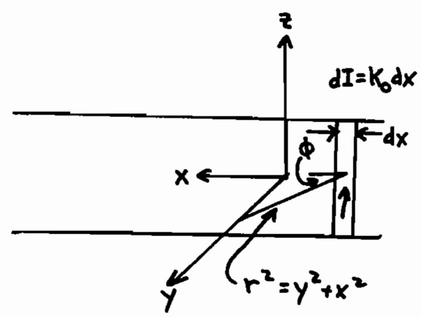


$$= \frac{\mu_o I}{4\pi (z^2 + r^2)^{3/2}} r dz \, \bar{i}_\phi$$

$$B_\phi = \frac{\mu_o I}{4\pi} \int_{z=-L}^{L} \frac{dz}{(z^2 + r^2)^{3/2}} = \frac{\mu_o I r}{4\pi} \left. \frac{z}{r^2 \sqrt{z^2 + r^2}} \right|_{z=-L}^{L} = \frac{\mu_o I L}{2\pi r \sqrt{L^2 + r^2}}$$

b) $B_y = 0$

$$dB_x = \frac{\mu_o L K_o dx}{2\pi \sqrt{y^2 + x^2} \sqrt{L^2 + y^2 + x^2}} \sin\phi \; ; \quad \sin\phi = \frac{y}{\sqrt{y^2 + x^2}}$$



$$= \frac{\mu_o L K_o y \, dx}{2\pi (y^2 + x^2)[L^2 + y^2 + x^2]^{1/2}}$$

$$B_x = \frac{\mu_o L K_o y}{2\pi} \int_{x=-\infty}^{+\infty} \frac{dx}{(y^2 + x^2)(L^2 + y^2 + x^2)^{1/2}}$$

Let $u = x^2 + y^2$, $du = 2xdx$, $x = \sqrt{u - y^2} \to du = 2\sqrt{u - y^2}\, dx$ $\qquad y > 0$

$$B_x = \frac{\mu_o L K_o y}{2\pi} \int_{y^2}^{\infty} \frac{du}{\sqrt{u - y^2}\, u \sqrt{L^2 + u}}$$

$$= \frac{\mu_o K_o L y}{2\pi} \int_{y^2}^{\infty} \frac{du}{u[u^2 + u(L^2 - y^2) - y^2 L^2]^{1/2}}$$

$$= \frac{\mu_o K_o L y}{2\pi} \frac{1}{yL} \sin^{-1} \left. \left[\frac{(L^2 - y^2)u - 2y^2 L^2}{u\sqrt{(L^2 - y^2)^2 + 4y^2 L^2}} \right] \right|_{u=y^2}^{\infty}$$

$$= \frac{\mu_o K_o}{2\pi}\left\{\sin^{-1}\frac{(L^2 - y^2)}{(L^2 + y^2)} - \sin^{-1}(-1)\right\}$$

$$= \frac{\mu_o K_o}{2\pi}\left\{\frac{\pi}{2} + \sin^{-1}\left(\frac{L^2 - y^2}{L^2 + y^2}\right)\right\} \qquad ; y > 0$$

Check:

$$\lim_{L\to\infty} B_x = \frac{\mu_o K_o}{2}$$

14. Use superposition

$$\overline{B}_1 = \begin{cases} \mu_o K_\phi \overline{i}_z & r < a \\ 0 & r > a \end{cases} \quad ; \quad \overline{B}_2 = \begin{cases} 0 & r < a \\ \dfrac{\mu_o K_z a}{r}\,\overline{i}_\phi & r > a \end{cases}$$

$$\overline{B}_T = \overline{B}_1 + \overline{B}_2 = \begin{cases} \mu_o K_o \cos\theta_o\, \overline{i}_z & r < a \\ \dfrac{\mu_o K_o a}{r}\sin\theta_o \overline{i}_\phi & r > a \end{cases}$$

15. a) Replace hole by currents $J_o\overline{i}_z$ and $-J_o\overline{i}_z$ and use superposition. Field due to slab with uniform current is given by (12) in (5.2.3b).

$$\overline{B}_1 = \begin{cases} -\dfrac{\mu_o J_o d}{2}\,\overline{i}_x & y > \dfrac{d}{2} \\ -\mu_o J_o y\,\overline{i}_x & |y| < \dfrac{d}{2} \\ \dfrac{\mu_o J_o d}{2}\,\overline{i}_x & y < -\dfrac{d}{2} \end{cases}$$

Field due to hole with current $-J_o\overline{i}_z$ [from (23) in (5.3.3b)]

$$\overline{B}_2 = \begin{cases} -\dfrac{\mu_o J_o a^2}{2r}\,\overline{i}_\phi = -\dfrac{\mu_o J_o a^2}{2[x^2+y^2]^{1/2}}(-\sin\phi\overline{i}_x + \cos\phi\overline{i}_y) \\ \qquad = -\dfrac{\mu_o J_o a^2}{2(x^2+y^2)}[-y\overline{i}_x + x\overline{i}_y] & r > a \\ -\dfrac{\mu_o J_o r}{2}\,\overline{i}_\phi = -\dfrac{\mu_o J_o}{2}[x^2+y^2]^{1/2}[-\sin\phi\overline{i}_x + \cos\phi\overline{i}_y] \end{cases}$$

$$= -\frac{\mu_o J_o}{2}[-y\bar{i}_x + x\bar{i}_y] \qquad r < a$$

$$\bar{B}_T = \bar{B}_1 + \bar{B}_2$$

b) Fill hole with current $J_o\bar{i}_z$

$$\bar{B}_1 = \frac{\mu_o J_o r}{2}\bar{i}_\phi = -\frac{\mu_o J_o}{2}[y\bar{i}_x - x\bar{i}_y] \qquad \text{(in hole)}$$

Fill hole with current $-J_o\bar{i}_z$

$$\bar{B}_2 = \frac{\mu_o J_o}{2}[y\bar{i}_x - (x-d)\bar{i}_y] \qquad \text{(in hole)}$$

$$\bar{B}_T = \bar{B}_1 + \bar{B}_2 = \frac{\mu_o J_o d}{2}\bar{i}_y$$

Section 5.3

16. a) $\bar{B} = ar\bar{i}_r$

$$\nabla\cdot\bar{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) = 3a \neq 0 \qquad \text{Not a magnetic field}$$

b) $\bar{B} = a(x\bar{i}_y - y\bar{i}_x)$

$$\nabla\cdot\bar{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \qquad \text{Can be a magnetic field}$$

$$\nabla\times\bar{B} = \bar{i}_z\left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right] = 2a\bar{i}_z = \mu_o\bar{J}$$

c) $\bar{B} = a(x\bar{i}_x - y\bar{i}_y)$

$$\nabla\cdot\bar{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \qquad \text{Can be a magnetic field}$$

$$\nabla\times\bar{B} = \bar{i}_z\left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right] = \mu_o\bar{J} = 0$$

d) $\overline{B} = ar\overline{i}_\phi$

$$\nabla \cdot \overline{B} = \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} = 0 \qquad \text{Can be a magnetic field}$$

$$\nabla \times \overline{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi)\overline{i}_z = 2a\overline{i}_z = \mu_o \overline{J}$$

17. a)
$$\frac{\partial B_x}{\partial y} = \begin{cases} -\mu_o J_o & -a < y < 0 \\ \mu_o J_o & 0 < y < a \end{cases}$$

$$B_x = \begin{cases} -\mu_o J_o (y + a) & -a < y < 0 \\ \mu_o J_o (y - a) & 0 < y < a \\ 0 & |y| > a \end{cases}$$

b)
$$\frac{\partial B_x}{\partial y} = \frac{-\mu_o J_o y}{a} \rightarrow B_x = \begin{cases} \frac{-\mu_o J_c}{2a}(y^2 - a^2) & |y| < a \\ 0 & |y| > a \end{cases}$$

c)
$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = \pm \mu_o J_o \rightarrow rB_\phi = \pm \frac{\mu_o J_o r^2}{2} + \text{constant}$$

$$B_\phi = \begin{cases} \frac{\mu_o J_o r}{2} & 0 < r < a \\ -\frac{\mu_o J_o r}{2}\left(1 - \frac{2a^2}{r^2}\right) & a < r < b \\ \frac{\mu_o J_o}{r}\left(a^2 - \frac{b^2}{2}\right) & r > b \end{cases}$$

d)
$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = \frac{\mu_o J_o r}{a} \rightarrow rB_\phi = \frac{\mu_o J_o r^3}{3a} + \text{constant}$$

$$B_\phi = \begin{cases} \frac{\mu_o J_o r^2}{3a} & r < a \\ \frac{\mu_o J_o a^2}{3r} & r > a \end{cases}$$

Section 5.4

18. a) $$A_z = \frac{-\mu_o K_o}{4\pi}\left\{\int_{-\infty}^{0}\{\ln[(x'-x)^2+(y-d)^2]-\ln[(x-x')^2+(y+d)^2]\}dx'\right\}$$

$$= \frac{-\mu_o K_o}{4\pi}\{(x'-x)\ln[(x'-x)^2+(y-d)^2]-2(x'-x)+2(y-d)\tan^{-1}(\frac{x'-x}{y-d})$$

$$-(x'-x)\ln[(x'-x)^2+(y+d)^2]+2(x'-x)-2(y+d)\tan^{-1}(\frac{x'-x}{y+d})\}\Bigg|_{x'=-\infty}^{0}$$

$$= \frac{-\mu_o K_o}{4\pi}\Big\{-x\{\ln[x^2+(y-d)^2]-\ln[x^2+(y+d)^2]\}$$

$$-2(y-d)[\tan^{-1}\frac{x}{y-d}+\frac{\pi}{2}]$$

$$+2(y+d)[\tan^{-1}\frac{x}{y+d}-\frac{\pi}{2}]\Big\} \qquad |y| < d$$

b) $$\overline{B} = \nabla\times\overline{A} = \overline{i}_x\frac{\partial A_z}{\partial y} - \overline{i}_y\frac{\partial A_z}{\partial x}$$

$$B_x = \frac{\partial A_z}{\partial y} = \frac{-\mu_o K_o}{4\pi}\Big\{-x\,[\frac{2(y-d)}{x^2+(y-d)^2}-\frac{2(y+d)}{x^2+(y+d)^2}]+2\,[-\tan^{-1}\frac{x}{y-d}-\pi+\tan^{-1}\frac{x}{y+d}]$$

$$+2(y-d)\frac{x}{x^2+(y-d)^2}-\frac{2(y+d)x}{(y+d)^2+x^2}\Big\}$$

$$= \frac{\mu_o K_o}{2\pi}[\tan^{-1}\frac{x}{y-d}-\tan^{-1}\frac{x}{y+d}+\pi]$$

$$B_y = -\frac{\partial A_z}{\partial x} = \frac{\mu_o K_o}{4\pi}\Big\{-\ln[x^2+(y-d)^2]+\ln[x^2+(y+d)^2]\ -2x^2[\frac{1}{x^2+(y-d)^2}-\frac{1}{x^2+(y+d)^2}]$$

$$-\frac{2(y-d)^2}{x^2+(y-d)^2}+\frac{(y+d)^2}{x^2+(y+d)^2}\Big\}$$

$$= \frac{-\mu_o K_o}{4\pi}\ln\frac{x^2+(y-d)^2}{x^2+(y+d)^2}$$

c) $$\Phi = \int_{\substack{y=-d\\x=0}}^{+d} B_x\,dy = \int_{\substack{y=-d\\x=0}}^{d}\frac{\partial A_z}{\partial y} = [A_z(y=d) - A_z(y=-d)]\Big|_{x=0}$$

$$A_z(x=0,y) = \frac{\mu_o K_o y}{2} \rightarrow \Phi = \mu_o K_o d$$

$$\Phi(x=-\infty) = \mu_o K_o (2d) \rightarrow \Phi \text{ through each current sheet} = \frac{\mu_o K_o d}{2}$$

d) $A_z(x=0,y=y_o) = \dfrac{\mu_o K_o y_o}{2}$

$$A_z(x=-\infty,y) = \mu_o K_o y \rightarrow A_z(x=-\infty,y) = A_z(x=0,y=y_o) \rightarrow y = \frac{y_o}{2} \text{ at } x = -\infty$$

19. a) $A_z = \dfrac{\mu_o I}{4\pi}\left\{\sinh^{-1}\dfrac{\frac{L}{2}-z}{r} + \sinh^{-1}\dfrac{\frac{L}{2}+z}{r}\right\}$

$$\nabla\cdot\bar{A} = \frac{\partial A_z}{\partial z} = \frac{\mu_o I}{4\pi}\left\{\frac{-1}{\sqrt{r^2+(\frac{L}{2}-z)^2}} + \frac{1}{\sqrt{r^2+(\frac{L}{2}+z)^2}}\right\} \neq 0$$

b) For a square loop

$$\bar{A} = \frac{\mu_o I}{4\pi}\left\{\bar{i}_z\left[\sinh^{-1}\frac{\frac{L}{2}-z}{[x^2+(y+\frac{L}{2})^2]^{1/2}} + \sinh^{-1}\frac{\frac{L}{2}+z}{[x^2+(y+\frac{L}{2})^2]^{1/2}} - \sinh^{-1}\frac{\frac{L}{2}-z}{[x^2+(y-\frac{L}{2})^2]^{1/2}}\right.\right.$$

$$\left. - \sinh^{-1}\frac{\frac{L}{2}+z}{[x^2+(y-\frac{L}{2})^2]^{1/2}}\right] + \bar{i}_y\left[\sinh^{-1}\frac{\frac{L}{2}-y}{[x^2+(z-\frac{L}{2})^2]^{1/2}}\right.$$

$$\left.\left. + \sinh^{-1}\frac{\frac{L}{2}+y}{[x^2+(z-\frac{L}{2})^2]^{1/2}} - \sinh^{-1}\frac{\frac{L}{2}-y}{[x^2+(z+\frac{L}{2})^2]^{1/2}} - \sinh^{-1}\frac{\frac{L}{2}+y}{[x^2+(z+\frac{L}{2})^2]^{1/2}}\right]\right\}$$

c) $\nabla\cdot\bar{A} = \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

$$= \frac{\mu_o I}{4\pi}\left\{\frac{-1}{[x^2+(y+\frac{L}{2})^2+(\frac{L}{2}-z)^2]^{1/2}} + \frac{1}{[x^2+(y+\frac{L}{2})^2+(\frac{L}{2}+z)^2]^{1/2}} + \frac{1}{[x^2+(y-\frac{L}{2})^2+(\frac{L}{2}-z)^2]^{1/2}}\right.$$

$$- \frac{1}{[x^2+(y-\frac{L}{2})^2+(\frac{L}{2}+z)^2]^{1/2}} - \frac{1}{[x^2+(z-\frac{L}{2})^2+(\frac{L}{2}-y)^2]^{1/2}} + \frac{1}{[x^2+(z-\frac{L}{2})^2+(\frac{L}{2}+y)^2]^{1/2}}$$

$$+ \frac{1}{[x^2+(z+\frac{L}{2})^2+(\frac{L}{2}-y)^2]^{1/2}} - \frac{1}{[x^2+(z+\frac{L}{2})^2+(\frac{L}{2}+y)^2]^{1/2}}\Big\}$$

$$= 0$$

20. $\overline{A} = \frac{\mu_o}{4\pi}\int_S \frac{\overline{K}dS}{r_{QP}}$, $\nabla^2\overline{A} = -\mu_o\overline{J} \rightarrow \nabla \times (\nabla \times \overline{A}) = \mu_o\overline{J}$

a) $\overline{A} = A_\phi(r)\overline{i}_\phi$, $\nabla \times \overline{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\overline{i}_z$

$$\nabla \times (\nabla \times \overline{A}) = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\overline{i}_\phi = 0 \rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) = C$$

$$rA_\phi = \frac{Cr^2}{2} + D$$

$$A_\phi = \begin{cases} \frac{C_1 r}{2} & 0 < r < a \\ \frac{D}{r} & r > a \end{cases}$$

$$\overline{B} = \nabla \times \overline{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\overline{i}_z = \begin{cases} C_1\overline{i}_z = \mu_o K_o\overline{i}_z & 0 < r < a \\ 0 & r > a \end{cases}$$

b) and c) $\overline{A} = A_z(r)\,\overline{i}_z$

$$\nabla \times \overline{A} = -\frac{\partial A_z}{\partial r}\overline{i}_\phi, \quad \nabla \times (\nabla \times \overline{A}) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = -\mu_o J_o$$

$$r\frac{\partial A_z}{\partial r} = \frac{-\mu_o J_o r^2}{2} + C \rightarrow A_z = \frac{-\mu_o J_o r^2}{4} + C\ln r + D$$

b) Surface current $K_o\overline{i}_z \rightarrow J_o = 0$

$$A_z = \begin{cases} D_1 & 0 < r < a \\ C\ln r + D_2 & r > a \end{cases}$$

$$\overline{B} = \nabla \times \overline{A} = -\frac{\partial A_z}{\partial r}\overline{i}_\phi = \begin{cases} 0 & 0 < r < a \\ -\frac{C}{r}\overline{i}_\phi = \frac{\mu_o K_o a}{r}\overline{i}_\phi & r > a \end{cases}$$

c) Volume current $J_o\bar{i}_z$

$$A_z = \begin{cases} \dfrac{-\mu_o J_o r^2}{4} + D_1 \\ C\ln r + D_2 \end{cases} ; \quad B_\phi = -\frac{\partial A_z}{\partial r} = \begin{cases} \dfrac{\mu_o J_o r}{2} & 0 < r < a \\ \dfrac{\mu_o J_o a^2}{2r} & r > a \end{cases}$$

d) $\bar{A} = A_z(x)\bar{i}_z$

$$\frac{\partial^2 A_z}{\partial x^2} = -\mu_o J_o \rightarrow A_z = \frac{-\mu_o J_o x^2}{2} + A_1 x + A_2$$

$$\bar{B} = \nabla \times \bar{A} = -\bar{i}_y \frac{\partial A_z}{\partial x} = (\mu_o J_o x - A_1)\bar{i}_y$$

$$\bar{B} = \begin{cases} \mu_o J_o x & |y| < \frac{d}{2} \\ \mu_o J_o \frac{d}{2} & y > \frac{d}{2} \\ -\mu_o J_o \frac{d}{2} & y < -\frac{d}{2} \end{cases}$$

e) $$\frac{\partial^2 A_z}{\partial x^2} = \frac{-\mu_o J_o x}{d} \rightarrow A_z = \frac{-\mu_o J_o x^3}{6d} + A_1 x + A_2$$

$$\bar{B} = \nabla \times \bar{A} = -\bar{i}_y \frac{\partial A_z}{\partial x} = \left(\frac{\mu_o J_o x^2}{2d} - A_1\right)\bar{i}_y \rightarrow B_y = \begin{cases} \dfrac{\mu_o J_o}{2d}\left(x^2 - \dfrac{d^2}{2}\right) & |x| < \frac{d}{2} \\ 0 & |x| > \frac{d}{2} \end{cases}$$

<u>Section 5.5</u>

21. a) $\bar{m} = \frac{1}{2}\bar{r} \times q\bar{v} = \frac{1}{2} q\omega a^2(\bar{i}_r \times \bar{i}_\phi) = \frac{1}{2} q\omega a^2\bar{i}_z$

b) $\bar{m} = \frac{1}{2}\oint a\bar{i}_r \times Iad\phi\bar{i}_\phi$

$= I\pi a^2\bar{i}_z$

c) $\bar{m} = \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{r=0}^{a} rK_o(\bar{i}_r \times \bar{i}_\phi)rdrd\phi$

$$= K_o \pi \bar{i}_z \int_{r=0}^{a} r^2 dr$$

$$= \frac{K_o \pi a^3}{3} \bar{i}_z$$

d) Surface charge $\sigma_f = \frac{Q}{4\pi R^2}$; $\bar{K} = \sigma_f \omega R \bar{i}_\phi = \frac{Q\omega}{4\pi R} \bar{i}_\phi$

$$\bar{m} = \frac{1}{2} \frac{Q\omega}{4\pi R} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} R^3 \underbrace{(\bar{i}_r \times \bar{i}_\phi)}_{-\bar{i}_\theta} \sin\theta d\theta d\phi$$

$$= -\frac{1}{2} \frac{Q\omega R^2}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} [\cos\theta\cos\phi \bar{i}_x + \cos\theta\sin\phi \bar{i}_y - \sin\theta \bar{i}_z] \sin\theta d\theta d\phi$$

$m_x = 0$, $m_y = 0$ when integrated over ϕ

$$m_z = \frac{1}{4} Q\omega R^2 \int_{\theta=0}^{\pi} \sin^2\theta d\theta$$

$$= Q\omega R^2 \frac{\pi}{8}$$

Volume charge $\rho_f = \frac{Q}{\frac{4}{3}\pi R^3}$; $\bar{J} = \rho_f \omega r \bar{i}_\phi = \frac{Q\omega}{\frac{4}{3}\pi R^3} r \bar{i}_\phi$

$$\bar{m} = \frac{1}{2} \frac{Q\omega}{\frac{4}{3}\pi R^3} \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^4 (\bar{i}_r \times \bar{i}_\phi) \sin\theta dr d\theta d\phi$$

$$m_z = \frac{3Q\omega}{4R^3} \int_{r=0}^{R} \int_{\theta=0}^{\pi} r^4 \sin^2\theta dr d\theta$$

$$= \frac{3Q\omega\pi R^2}{40}$$

22. a) $\bar{H} = [H_o + \frac{m}{2\pi |z|^3}] \bar{i}_z$ along z axis

b) $f_z = \mu_o m_z \frac{\partial H_z}{\partial z} = \mu_o m [\frac{-3m}{2\pi z^4}] = -\frac{3\mu_o m^2}{2\pi z^4}$

c) $H_z = H_o + \frac{m}{2\pi}\sum_{n=1}^{\infty}\frac{1}{(na)^3} \approx H_o + \frac{1.2m}{2\pi a^3} \approx H_o + \frac{1.2}{2\pi a^3}\alpha H_z$

$$H_z \approx \frac{H_o}{1 - \frac{1.2\alpha}{2\pi a^3}}, \quad m_z = \alpha H_z = \frac{\alpha H_o}{1 - \frac{1.2\alpha}{2\pi a^3}}$$

d) $B_z = \mu_o(H_o + \frac{m_z}{a^3})$

$$= \mu_o H_o\left[1 + \frac{\alpha}{a^3[1 - \frac{1.2\alpha}{2\pi a^3}]}\right] \rightarrow \mu = \mu_o\left[1 + \frac{\alpha}{a^3[1 - \frac{1.2\alpha}{2\pi a^3}]}\right]$$

23. a) $\frac{d\overline{L}}{dt} = \overline{m} \times \overline{B}, \quad \overline{L} = -\frac{2m_e}{e}\overline{m} \rightarrow \frac{d\overline{m}}{dt} = -\gamma\overline{m} \times \overline{B}; \quad \gamma = \frac{e}{2m_e}$

b) $\frac{dm_x}{dt} = -\gamma m_y B_o, \quad \frac{dm_y}{dt} = \gamma m_x B_o, \quad \frac{dm_z}{dt} = 0$

$$\frac{d^2m_x}{dt^2} + \omega_o^2 m_x = 0; \quad \omega_o^2 = \gamma B_o$$

$$m_x = m_{yo}\sin\omega_o t + m_{xo}\cos\omega_o t$$

$$m_y = m_{yo}\cos\omega_o t - m_{xo}\sin\omega_o t$$

$$m_z = m_{zo}$$

c) Precessional frequency $= \omega_o$.

24. a) $\oint_L \overline{H}\cdot\overline{d\ell} = \int_S \overline{J_f}\cdot\overline{dS} \rightarrow H_\phi(2\pi r) = \begin{cases} J_o\pi r^2 & r < a \\ J_o\pi a^2 & r > a \end{cases}$

$$H_\phi = \begin{cases} \frac{J_o r}{2} \\ \frac{J_o a^2}{2r} \end{cases}; \quad B_\phi = \begin{cases} \frac{\mu J_o r}{2} \\ \frac{\mu_o J_o a^2}{2r} \end{cases}; \quad M_\phi = \frac{B_\phi}{\mu_o} - H_\phi = \begin{cases} \frac{(\mu - \mu_o)}{\mu_o}\frac{J_o r}{2} & r < a \\ 0 & r > a \end{cases}$$

$$\overline{J}_m = \nabla \times \overline{M} = \frac{\mu - \mu_o}{\mu_o} J_o \overline{i}_z$$

$$K_{mz} = - M_\phi(r=a_-) = -(\frac{\mu - \mu_o}{\mu_o})\frac{J_o a}{2}$$

b) $$H_x = \begin{cases} -\frac{K_o}{2} & y > 0 \\ +\frac{K_o}{2} & y < 0 \end{cases} \; ; \; B_x = \begin{cases} -\frac{\mu K_o}{2} & 0 < y < \frac{d}{2} \\ \frac{\mu K_o}{2} & -\frac{d}{2} < y < 0 \\ -\frac{\mu_o K_o}{2} & y > \frac{d}{2} \\ \frac{\mu_o K_o}{2} & y < -\frac{d}{2} \end{cases} \; ; \; M_x = \begin{cases} -\frac{(\mu-\mu_o)K_o}{2\mu_o} & 0 < y < \frac{d}{2} \\ \frac{(\mu-\mu_o)K_o}{2\mu_o} & -\frac{d}{2} < y < 0 \\ 0 & |y| > \frac{d}{2} \end{cases}$$

$$K_{zm}(y=0) = M_x(y=0_+) - M_x(y=0_-) = -\frac{(\mu - \mu_o)K_o}{\mu_o}$$

$$K_{zm}(y=\frac{d}{2}) = M_x(y=\frac{d_+}{2}) - M_x(y=\frac{d_-}{2}) = \frac{\mu - \mu_o}{2\mu_o} K_o$$

$$K_{zm}(y=-\frac{d}{2}) = M_x(y=-\frac{d_+}{2}) - M_x(y=-\frac{d_-}{2}) = \frac{\mu - \mu_o}{2\mu_o} K_o$$

Section 5.6

25. $\overline{n}\cdot[\overline{B}_2 - \overline{B}_1] = 0 \rightarrow \mu_1 H_1 \cos\theta_1 = \mu_2 H_2 \cos\theta_2$

$\overline{n} \times [\overline{H}_2 - \overline{H}_1] = 0 \rightarrow H_1 \sin\theta_1 = H_2 \sin\theta_2$

$$\tan\theta_2 = \frac{\mu_2}{\mu_1}\tan\theta_1$$

$$H_2 = \frac{\mu_1}{\mu_2} H_1 \frac{\cos\theta_1}{\cos\theta_2} = \frac{\mu_1}{\mu_2} H_1 \cos\theta_1 \sqrt{1 + \tan^2\theta_2} = \frac{\mu_1}{\mu_2} H_1 \cos\theta_1 \sqrt{1 + (\frac{\mu_2}{\mu_1}\tan\theta_1)^2}$$

26. Replace magnetization by surface current at $r = a$, $K_\phi = M_o$.

a) From (5.2.4c)

$$B_z = \frac{\mu_o M_o}{2}\left\{\frac{\frac{L}{2} - z}{[(z - \frac{L}{2})^2 + a^2]^{1/2}} + \frac{\frac{L}{2} + z}{[(z + \frac{L}{2})^2 + a^2]^{1/2}}\right\}$$

$$H_z = \begin{cases} \dfrac{B_z}{\mu_o} - M_o & |z| < \dfrac{L}{2} \\[2ex] \dfrac{B_z}{\mu_o} & |z| > \dfrac{L}{2} \end{cases}$$

b) $$\lim_{z \gg \frac{L}{2}} B_z = \frac{\mu_o M_o}{2} \left\{ \frac{-1}{\left[1 + \frac{a^2}{(z-\frac{L}{2})^2}\right]^{1/2}} + \frac{1}{\left[1 + \frac{a^2}{(z+\frac{L}{2})^2}\right]^{1/2}} \right\}$$

$$= \frac{\mu_o M_o}{2} \left\{ -1 + \frac{1}{2} \frac{a^2}{(z - \frac{L}{2})^2} + 1 - \frac{1}{2} \frac{a^2}{(z + \frac{L}{2})^2} \right\}$$

$$= \frac{\mu_o M_o a^2}{4z^2} \left\{ \frac{1}{(1 - \frac{L}{2z})^2} - \frac{1}{(1 + \frac{L}{2z})^2} \right\}$$

$$= \frac{\mu_o M_o a^2}{4z^2} \left\{ 1 + \frac{L}{z} - (1 - \frac{L}{z}) \right\}$$

$$= \frac{\mu_o M_o a^2 L}{2z^3}$$

Far field of dipole along z axis

$$\lim_{z \gg \frac{L}{2}} B_z = \frac{\mu_o m}{2\pi z^3} \rightarrow m = M_o \pi a^2 L$$

c) $$\lim_{a \to \infty} B_z = 0, \quad H_z = \begin{cases} -M_o & |z| < \dfrac{L}{2} \\[2ex] 0 & |z| > \dfrac{L}{2} \end{cases}$$

d) $$\bar{J}_m = \nabla \times \bar{M} = -\frac{\partial M_z}{\partial r} \bar{i}_\phi = \frac{M_o}{a} \bar{i}_\phi$$

$$dK_\phi = \frac{M_o}{a} dr'$$

From (a)

$$dB_z = \frac{\mu_o M_o}{2a} dr' \left\{ \frac{\frac{L}{2} - z}{[(z - \frac{L}{2})^2 + r'^2]^{1/2}} + \frac{\frac{L}{2} + z}{[(z + \frac{L}{2})^2 + r'^2]^{1/2}} \right\}$$

$$B_z = \int_{r=0}^{a} dB_z = \frac{\mu_o M_o}{2a}\left\{(\frac{L}{2} - z)\ln[r'+\sqrt{r'^2+(z-\frac{L}{2})^2}] + (\frac{L}{2}+z)\ln[r'+\sqrt{r'^2+(z+\frac{L}{2})^2}]\right\}\Bigg|_{r=0}^{a}$$

$$= \frac{\mu_o M_o}{2a}\left\{(\frac{L}{2} - z)\ln\frac{a+\sqrt{a^2+(z-\frac{L}{2})^2}}{|z - \frac{L}{2}|} + (\frac{L}{2}+z)\ln\frac{a+\sqrt{a^2+(z+\frac{L}{2})^2}}{|z + \frac{L}{2}|}\right\}$$

Section 5.7

27. a) $\overline{H}_1 = \frac{I}{2\pi}\frac{[-y\overline{i}_x + (x-d)\overline{i}_y]}{[(x-d)^2 + y^2]} + \frac{I'}{2\pi}\frac{[-y\overline{i}_x + (x+d)\overline{i}_y]}{[(x+d)^2 + y^2]}$

$$\overline{H}_2 = \frac{I''}{2\pi}\frac{[-y\overline{i}_x + (x-d)\overline{i}_y]}{[(x-d)^2 + y^2]}$$

Boundary Conditions:

At $x = 0$: $\mu_1 H_{1x} = \mu_2 H_{2x} \rightarrow \mu_1(I + I') = \mu_2 I''$ $\qquad I' = \frac{(\mu_2 - \mu_1)I}{\mu_1 + \mu_2}$

$\rightarrow$

$H_{1y} = H_{2y} \qquad -I + I' = -I'' \qquad I'' = \frac{2\mu_1 I}{\mu_1 + \mu_2}$

b) $\overline{f} = -\frac{\mu_1 I\, I'}{2\pi(2d)}\overline{i}_x = -\frac{\mu_1(\mu_2 - \mu_1)I^2}{(\mu_1 + \mu_2)4\pi d}\overline{i}_x$

28. a) Try image $-I$ at $b = a^2/D$ from center and current $I_o + I$ at center.

$$\overline{H} = \frac{I}{2\pi}\frac{[-y\overline{i}_x+(x+D)\overline{i}_y]}{[(x+D)^2 + y^2]} - \frac{I}{2\pi}\frac{[-y\overline{i}_x+(x+\frac{a^2}{D})\overline{i}_y]}{[y^2 + (x+\frac{a^2}{D})^2]} + \frac{(I_o+I)[-y\overline{i}_x+x\overline{i}_y]}{2\pi[x^2 + y^2]}$$

$$x\overline{i}_y - y\overline{i}_x = r\overline{i}_\phi; \quad \overline{i}_y = \sin\phi\,\overline{i}_r + \cos\phi\,\overline{i}_\phi$$

$$\overline{H} = \frac{I}{2\pi}\frac{[r\overline{i}_\phi + D\overline{i}_y]}{[(x+D)^2+y^2]} - \frac{I}{2\pi}\frac{[r\overline{i}_\phi + \frac{a^2}{D}\overline{i}_y]}{[y^2+(x+\frac{a^2}{D})^2]} + \frac{(I_o + I)}{2\pi r}\overline{i}_\phi$$

Check: $H_r(x^2 + y^2 = a^2) = 0$

$$H_r = \frac{I}{2\pi}D\sin\phi\left[\frac{1}{[(x + D)^2 + y^2]} - \frac{a^2/D^2}{[y^2 + (x + a^2/D)^2]}\right]$$

$$= \frac{ID}{2\pi}\sin\phi\left[\frac{1}{x^2+y^2+2Dx+D^2} - \frac{1}{\frac{D^2}{a^2}\left[x^2+y^2+\frac{2xa^2}{D}+\left(\frac{a^2}{D}\right)^2\right]}\right]$$

$$H_r(x^2+y^2=a^2) = 0$$

a) and b) $$H_\phi = \frac{Ir}{2\pi}\left|\frac{1}{[(x+D)^2+y^2]} - \frac{1}{[y^2+(x+\frac{a^2}{D})^2]}\right|$$

$$+\frac{ID}{2\pi}\cos\phi\left\{\frac{1}{[(x+D)^2+y^2]} - \frac{a^2/D^2}{[y^2+(x+\frac{a^2}{D})^2]}\right\} + \frac{I_o+I}{2\pi r}$$

$$H_\phi(x^2+y^2=a^2) = \frac{Ia}{2\pi}\left\{\frac{1}{a^2+D^2+2xD} - \frac{1}{a^2+(\frac{a^2}{D})^2+\frac{2xa^2}{D}}\right\} + \frac{(I_o+I)}{2\pi a}$$

$$x = a\cos\phi$$

$$K_z(\phi) = H_\phi(x^2+y^2=a^2) = \frac{Ia}{2\pi}\left\{\frac{1}{a^2+D^2+2aD\cos\phi} - \frac{1}{[a^2+D^2+2aD\cos\phi]\frac{a^2}{D^2}}\right\} + \frac{(I_o+I)}{2\pi a}$$

$$I_z = 2\int_{\phi=0}^{\pi} K_z a\,d\phi$$

$$= \frac{Ia^2}{\pi}\int_{\phi=0}^{\pi}\frac{(1-\frac{D^2}{a^2})}{a^2+D^2+2aD\cos\phi}d\phi + (I_o+I)$$

$$= \frac{Ia^2}{\pi}\left(1-\frac{D^2}{a^2}\right)\frac{2}{[(a^2+D^2)^2-(2aD)^2]^{1/2}}\tan^{-1}\left\{\frac{[(a^2+D^2)^2-(2aD)^2]^{1/2}\tan\frac{\phi}{2}}{a^2+D^2+2aD}\right\}\Bigg|_{\phi=0}^{\pi} + (I_o+I)$$

$$= \frac{Ia^2(1-\frac{D^2}{a^2})}{[(a^2+D^2)^2-(2aD)^2]^{1/2}} + (I_o+I)$$

$$= \frac{I(a^2-D^2)}{[(a^2-D^2)^2]^{1/2}} + (I_o+I)$$

$$= I_o$$

Integration must be $0 < \phi < \pi$ and not $0 < \phi < 2\pi$ to avoid ambiguity in the principal value of inverse tangent function. The integrand is even in ϕ so that the integral for $\pi < \phi < 2\pi$ is the same as for $0 < \phi < \pi$ so that we only

integrate $0 < \phi < \pi$ and multiply the result by 2.

c) $$f_x = \frac{-I}{2\pi}\left[\frac{-I}{D - \frac{a^2}{D}} + \frac{(I_o + I)}{D}\right]$$

$$= \frac{I^2 D}{2\pi(D^2 - a^2)} - \frac{I(I + I_o)}{2\pi D} = \frac{I^2}{2\pi}\left[\frac{D}{D^2 - a^2} - \frac{1}{D}\right] - \frac{II_o}{2\pi D}$$

$$= \frac{I^2 a^2}{2\pi D(D^2 - a^2)} - \frac{II_o}{2\pi D}$$

d) Place image current $-I$ at $y = -y_o$ so that cylinder image current is at position

$$y = d - \frac{a^2}{d + y_o} = y_o$$ [to keep magnetic field tangential to cylinder and plane]

$$y_o(d + y_o) = d(d + y_o) - a^2$$

$$y_o^2 = d^2 - a^2 \rightarrow y_o = \sqrt{d^2 - a^2}$$

e) $$f_y = \frac{\mu_o I^2}{2\pi(2y_o)} = \frac{\mu_o I^2}{4\pi\sqrt{d^2 - a^2}}$$

29. a) $$\chi = \begin{cases} A_1 \sin ay\, e^{-ax} \\ A_2 \sin ay\, e^{+ax} \end{cases} ; \quad \overline{H} = \nabla\chi = \begin{cases} A_1 a[\cos ay\,\overline{i}_y - \sin ay\,\overline{i}_x]e^{-ax} & x > 0 \\ A_2 a[\cos ay\,\overline{i}_y + \sin ay\,\overline{i}_x]e^{+ax} & x < 0 \end{cases}$$

b) <u>Boundary Conditions</u>:

$$H_y(x=0_+) - H_y(x=0_-) = K_o \cos ay \rightarrow a(A_1 - A_2) = K_o$$

$$H_x(x=0_+) = H_x(x=0_-) \rightarrow -A_1 a = A_2 a \rightarrow A_1 = -A_2 = \frac{K_o}{2a}$$

c) $$\overline{B} = \nabla \times \overline{A} = \overline{i}_x \frac{\partial A_z}{\partial y} - \overline{i}_y \frac{\partial A_z}{\partial x} = \begin{cases} \frac{\mu_o K_o}{2}[\cos ay\,\overline{i}_y - \sin ay\,\overline{i}_x]e^{-ax} & x > 0 \\ -\frac{\mu_o K_o}{2}[\cos ay\,\overline{i}_y + \sin ay\,\overline{i}_x]e^{ax} & x < 0 \end{cases}$$

$$A_z = \begin{cases} \dfrac{\mu_o K_o}{2a} \cos ay e^{-ax} & x > 0 \\ \\ \dfrac{\mu_o K_o}{2a} \cos ay e^{+ax} & x < 0 \end{cases}$$

d) A_z = constant is equation of field lines.

30. a) Particular solution, $\bar{A} = A_z(x)\,\bar{i}_z$

$$\nabla^2 \bar{A} = -\mu_o \bar{J} \rightarrow \frac{d^2 A_z}{dx^2} = -\mu_o J_o \sin ax \rightarrow A_z = \begin{cases} \dfrac{\mu_o J_o}{a^2} \sin ax & 0 < y < d \\ \\ 0 & y > d \end{cases}$$

b) Homogeneous, $\bar{A} = A_z(x,y)\,\bar{i}_z$

$$\nabla^2 \bar{A} = 0 \rightarrow \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = 0 \rightarrow A_z = \sin ax[A_1 e^{ay} + A_2 e^{-ay}]$$

$$A_{z\,total} = \begin{cases} \sin ax \left[\dfrac{\mu_o J_o}{a^2} + A_1 e^{ay} + A_2 e^{-ay}\right] & 0 < y < d \\ \\ \sin ax B_1 e^{-a(y - d)} & y > d \end{cases}$$

$$\bar{B} = \nabla \times \bar{A} = \bar{i}_x \frac{\partial A_z}{\partial y} - \bar{i}_y \frac{\partial A_z}{\partial x}$$

$$= \begin{cases} \bar{i}_x a \sin ax[A_1 e^{ay} - A_2 e^{-ay}] - \bar{i}_y a \cos ax \left[\dfrac{\mu_o J_o}{a^2} + A_1 e^{ay} + A_2 e^{-ay}\right] & 0<y<d \\ \\ -B_1 a[\bar{i}_x \sin ax + \bar{i}_y \cos ax] e^{-a(y - d)} & y > d \end{cases}$$

<u>Boundary Conditions:</u>

$$B_y(y=0) = 0 \rightarrow A_1 + A_2 + \frac{\mu_o J_o}{a^2} = 0$$

$$B_x(y=d_-) = B_x(y=d_+) \rightarrow -B_1 = A_1 e^{ad} - A_2 e^{-ad}$$

$$B_y(y=d_-) = B_y(y=d_+) \rightarrow B_1 = \frac{\mu_o J_o}{a^2} + A_1 e^{ad} + A_2 e^{-ad}$$

$$A_1 = -\frac{\mu_o J_o}{2a^2} e^{-ad}, \quad A_2 = -\frac{\mu_o J_o}{a^2}\left(1 - \frac{e^{-ad}}{2}\right), \quad B_1 = \frac{\mu_o J_o}{2a^2}(1 - e^{-ad})^2$$

c) $K_z(y=0) = -H_x(y=0) = -a\sin ax[A_1 - A_2] = \frac{J_o}{a}\sin ax(e^{-ad} - 1)$

d) $F_{Ay} = \frac{1}{2}\mu_o K_z(y=0) \times H_x(y=0) = -\frac{1}{2}\mu_o K_z^2(y=0) = -\frac{\mu_o J_o^2}{2a^2}\sin^2 ax(1 - e^{-ad})^2$

$$f_{Ay} = \int_0^{2\pi/a} F_{Ay}dx$$

$$= -\frac{\mu_o J_o^2}{2a^2}(1 - e^{-ad})^2 \int_0^{2\pi/a} \sin^2 ax dx$$

$$= -\frac{\mu_o J_o^2 \pi}{2a^3}(1 - e^{-ad})^2$$

$$\overline{F}_V = \overline{J} \times \overline{B}$$

$$= J_o \sin ax\, \overline{i}_z \times [B_x\overline{i}_x + B_y\overline{i}_y]$$

$$= J_o \sin ax[B_x\overline{i}_y - B_y\overline{i}_x]$$

$$f_{Vy} = J_o a \int_{y=0}^{d}\int_{x=0}^{2\pi/a} [A_1 e^{ay} - A_2 e^{-ay}]\sin^2 ax dx dy$$

$$= \frac{J_o \pi}{a}[A_1 e^{ay} + A_2 e^{-ay}]\Big|_{y=0}^{d}$$

$$= \frac{-J_o^2 \pi\mu_o}{2a^3}[e^{a(y-d)} - e^{-a(y+d)} + 2e^{-ay}]\Big|_{y=0}^{d}$$

$$= \frac{-\mu_o \pi J_o^2}{2a^3}[1 - e^{-2ad} + 2e^{-ad} - e^{-ad} + e^{-ad} - 2]$$

$$= \frac{-\mu_o \pi J_o^2}{2a^3}[-1 - e^{-2ad} + 2e^{-ad}]$$

$$= \frac{\mu_o \pi J_o^2}{2a^3}(1 - e^{-ad})^2$$

$$f_{Ay} + f_{Vy} = 0$$

e) $\overline{M} = M_o \sin ax\overline{i}_y \rightarrow \nabla\cdot\overline{M} = 0, \quad \nabla\cdot\overline{B} = \nabla\cdot\overline{H} = 0, \quad \nabla \times \overline{H} = 0 \rightarrow \overline{H} = \nabla\chi, \quad \nabla^2\chi = 0$

$$\chi = \begin{cases} \sin ax[A_1e^{ay} + A_2e^{-ay}] & 0 < y < d \\ B_1 \sin ax\ e^{-a(y-d)} & y > d \end{cases}$$

$$\overline{H} = \nabla\chi = \begin{cases} a[\overline{i}_x \cos ax(A_1e^{ay}+A_2e^{-ay}) + \overline{i}_y \sin ax(A_1e^{ay}-A_2e^{-ay})] & 0 < y < d \\ B_1 a[\overline{i}_x \cos ax - \overline{i}_y \sin ax]e^{-a(y-d)} & y > d \end{cases}$$

Boundary Conditions:

$$B_y(y=0) = \mu_o[H_y(y=0) + M_o \sin ax] = 0 \rightarrow \frac{M_o}{a} + A_1 - A_2 = 0$$

$$H_x(y=d_+) = H_x(y=d_-) \rightarrow B_1 = A_1e^{ad} + A_2e^{-ad}$$

$$B_y(y=d_+) = B_y(y=d_-) \rightarrow -B_1 = A_1e^{ad} - A_2e^{-ad} + \frac{M_o}{a}$$

$$A_1 = -\frac{M_o}{2a}e^{-ad}, \quad A_2 = \frac{M_o}{a}\left(1 - \frac{e^{-ad}}{2}\right), \quad B_1 = -\frac{M_o}{2a}(1 - e^{-ad})^2$$

$$K_z(y=0) = -H_x(y=0) = -a\cos ax(A_1 + A_2) = -M_o \cos ax(1 - e^{-ad})$$

$$F_{Ay} = \frac{1}{2}\mu_o\ K_z(y=0)H_x(y=0) = -\frac{1}{2}\mu_o K_z^2(y=0) = -\frac{M_o^2}{2}\mu_o \cos^2 ax(1 - e^{-ad})^2$$

$$f_{Ay} = \int_{x=0}^{2\pi/a} F_{Ay}dx = -\frac{\mu_o M_o^2}{2}(1 - e^{-ad})^2 \int_{x=0}^{2\pi/a} \cos^2 axdx = -\frac{\mu_o M_o^2 \pi}{2a}(1 - e^{-ad})^2$$

$$\overline{F}_V = \mu_o(\overline{M}\cdot\nabla)\overline{H} = \mu_o M_o \sin ax \frac{\partial}{\partial y}[H_x\overline{i}_x + H_y\overline{i}_y]$$

$$F_{Vy} = \mu_o M_o \sin^2 ax\ a^2[A_1e^{ay} + A_2e^{-ay}]$$

$$f_{Vy} = \mu_o M_o a^2 \int_{y=0}^{d}\int_{x=0}^{2\pi/a} \sin^2 ax[A_1e^{ay} + A_2e^{-ay}]dxdy$$

$$= \mu_o M_o \pi[A_1e^{ay} - A_2e^{-ay}]\Big|_{y=0}^{d}$$

$$= \mu_o M_o \pi [A_1(e^{ad} - 1) - A_2(e^{-ad} - 1)]$$

$$= \frac{\mu_o M_o^2 \pi}{2a}(1 - 2e^{-ad} + e^{-2ad}) = \frac{\mu_o M_o^2 \pi}{2a}(1 - e^{-ad})^2 = -f_{Ay}$$

31. a)

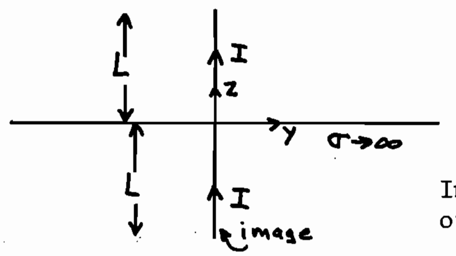


Image current in the same direction as original current.

b) $\overline{B} = \dfrac{\mu_o I}{4\pi r}\left\{\dfrac{L - z}{[r^2 + (L - z)^2]^{1/2}} + \dfrac{L + z}{[r^2 + (L + z)^2]^{1/2}}\right\}\overline{i}_\phi$

c) $\overline{K}(z=0) = -\overline{i}_z \times \overline{H}(z=0) = \dfrac{-I\ L}{2\pi r[r^2 + (L)^2]^{1/2}}\,\overline{i}_r$

32. $\nabla^2\chi = 0,\ \overline{H} = \nabla\chi$

$$\chi = \begin{cases} Ar\cos\phi & r \le a \\ (Br + \frac{C}{r})\cos\phi & r \ge a \end{cases}$$

$$\overline{H} = \nabla\chi = \begin{cases} A[\cos\phi\,\overline{i}_r - \sin\phi\,\overline{i}_\phi] & r < a \\ [B - \frac{C}{r^2}]\cos\phi\,\overline{i}_r - [B + \frac{C}{r^2}]\sin\phi\,\overline{i}_\phi & r > a \end{cases}$$

Boundary Conditions:

$$\overline{H}(r\to\infty) = H_o\overline{i}_x = H_o[\overline{i}_r\cos\phi - \overline{i}_\phi\sin\phi] \rightarrow B = H_o$$

a) $H_\phi(r=a_+) = H_\phi(r=a_-) \rightarrow A = B + \dfrac{C}{a^2}$ $\qquad\qquad A = \dfrac{2\mu_1 H_o}{\mu_1 + \mu_2}$

$\rightarrow$

$\mu_1 H_r(r=a_+) = \mu_2 H_r(r=a_-) \rightarrow \mu_2 A = \mu_1(B - \dfrac{C}{a^2})$ $\qquad\qquad C = \dfrac{H_o a^2(\mu_1 - \mu_2)}{\mu_1 + \mu_2}$

b) $A = 0, \quad H_r(r=a_+) = 0 \qquad B = \dfrac{C}{a^2} = H_o$

$$\overline{H} = \begin{cases} 0 & r < a \\ H_o\{[1 - \dfrac{a^2}{r^2}]\cos\phi\overline{i}_r - [1 + \dfrac{a^2}{r^2}]\sin\phi\overline{i}_\phi\} & r > a \end{cases}$$

c) $H_\phi(r=a_+) = H_\phi(r=a_-) \rightarrow A = B + \dfrac{C}{a^2}$

$H_r(r=a_+) + M_1\cos\phi = H_r(r=a_-) + M_2\cos\phi \rightarrow B - \dfrac{C}{a^2} + M_1 = A + M_2$

$$\rightarrow \quad A = H_o + \frac{(M_1 - M_2)}{2} \qquad C = a^2\,\frac{(M_1 - M_2)}{2}$$

$$\overline{H} = \begin{cases} [H_o + \dfrac{M_1 - M_2}{2}][\overline{i}_r\cos\phi - \overline{i}_\phi\sin\phi] = [H_o + \dfrac{M_1 - M_2}{2}]\,\overline{i}_x & r < a \\ [H_o - \dfrac{(M_1 - M_2)a^2}{2r^2}]\cos\phi\overline{i}_r - [H_o + \dfrac{(M_1 - M_2)a^2}{2r^2}]\sin\phi\overline{i}_\phi & r > a \end{cases}$$

33. a) $K(y) = \dfrac{4K_o}{\pi} \displaystyle\sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} \dfrac{\sin\frac{n\pi y}{d}}{n}$

b) $$\chi = \begin{cases} A_1\cos ky\, e^{-kx} & x > 0 \\ A_2\cos ky\, e^{+kx} & x < 0 \end{cases}$$

$$\overline{H} = \nabla\chi = \begin{cases} A_1 k[-\sin ky\overline{i}_y - \cos ky\overline{i}_x]e^{-kx} & x > 0 \\ A_2 k[-\sin ky\overline{i}_y + \cos ky\overline{i}_x]e^{+kx} & x < 0 \end{cases}$$

$$H_x(x=0_+) = H_x(x=0_-) \rightarrow A_1 = -A_2$$

$$H_y(x=0_+) - H_y(x=0_-) = K_n \sin\frac{n\pi y}{d} \rightarrow \frac{n\pi}{d}[-A_1 + A_2] = K_n; \qquad k_n = \frac{n\pi}{d}$$

$$A_1 = -A_2 = \frac{-K_n d}{2n\pi}; \qquad K_n = \frac{4K_o}{n\pi}$$

c) $$\overline{H} = \begin{cases} \sum\limits_{n\ odd} \dfrac{2K_o}{n\pi} [\sin \dfrac{n\pi y}{d} \overline{i}_y + \cos \dfrac{n\pi y}{d} \overline{i}_x] e^{-n\pi x/d} & x > 0 \\ \sum\limits_{n\ odd} \dfrac{2K_o}{n\pi} [-\sin \dfrac{n\pi y}{d} \overline{i}_y + \cos \dfrac{n\pi y}{d} \overline{i}_x] e^{+n\pi x/d} & x < 0 \end{cases}$$

d) $$K_z(y=0) = -H_x(y=0) = \begin{cases} \sum\limits_{n\ odd} \dfrac{-2K_o}{n\pi} e^{-n\pi x/d} & x > 0 \\ \sum\limits_{n\ odd} \dfrac{-2K_o}{n\pi} e^{+n\pi x/d} & x < 0 \end{cases}$$

$$K_z(y=d) = H_x(y=d) = K_z(y=0)$$

On $y = 0$ for $x > 0$

$$I_z = \int_{x=0}^{\infty} K_z(x)dx$$

$$= \frac{-2K_o}{\pi} \int_0^{\infty} \sum_{n\ odd} \frac{1}{n} e^{-n\pi x/d} dx$$

$$= \frac{-2K_o}{\pi} \sum_{n\ odd} \frac{d}{n^2 \pi} e^{-n\pi x/d} \Big|_{x=0}^{\infty}$$

$$= \frac{2K_o d}{\pi^2} \sum_{n\ odd} \frac{1}{n^2} = \frac{K_o d}{4}$$

$$I_z(x>0,\ y=0) = I_z(x<0,\ y=0) = I_z(x>0,\ y=d) = I_z(x<0,\ y=d) = \frac{K_o d}{4}$$

Section 5.8

34. a) From Prob. (32c) with $M_1 = 0$, $M_2 = M_o$, $H_o = 0$

$$\overline{H} = \begin{cases} -\dfrac{M_o}{2}[\overline{i}_r \cos\phi - \overline{i}_\phi \sin\phi] = -\dfrac{M_o}{2}\overline{i}_x & r < a \\ \dfrac{M_o a^2}{2r^2}[\cos\phi \overline{i}_r + \sin\phi \overline{i}_\phi] & r > a \end{cases}$$

b) $$\overline{f}_L = -I\overline{i}_z \times \mu_o \overline{H}(\phi = -90°) \qquad (I \text{ at } y = -b)$$

$$= \frac{\mu_o I M_o a^2}{2b^2} (\overline{i}_z \times \overline{i}_\phi)$$

$$= \frac{\mu_o I M_o a^2}{2b^2}\,\overline{i}_y$$

$$\overline{f}_L = -I\overline{i}_z \times \mu_o \overline{H}(\phi = 0°)$$

$$= \frac{-\mu_o I M_o a^2}{2b^2}(\overline{i}_z \times \overline{i}_r) = \frac{-\mu_o I M_o a^2}{2b^2}\,\overline{i}_y$$

35. a) $J_y = \frac{I}{Dd} \rightarrow \nabla \times \overline{H} = -\frac{dH_z}{dx}\overline{i}_y = J_y\overline{i}_y \rightarrow H_z(x) = \frac{-I}{Dd}x + \text{constant}$

$$H_z(x=0) = \frac{I}{D},\quad H_z(x=d) = 0 \rightarrow H_z(x) = \frac{-I}{Dd}(x - d)$$

b) $\overline{F} = \mu_o(\overline{M}\cdot\nabla)\overline{H} + \mu_o\overline{J}_f \times \overline{H}$

$$= \mu_o M_z \overset{0}{\frac{d\overline{H}}{dz}} + \frac{\mu_o I}{Dd}\,\overline{i}_y \times [H_z(x)\overline{i}_z]$$

$$F_x = -\frac{\mu_o I^2}{(Dd)^2}(x - d)$$

$$f_x = \int_{x=0}^{d} F_x sD dx$$

$$= -\frac{\mu_o I^2 s}{Dd^2}\int_{x=0}^{d}(x - d)dx$$

$$= -\frac{\mu_o I^2 s}{Dd^2}\frac{(x - d)^2}{2}\Bigg|_{x=0}^{d}$$

$$= \frac{1}{2}\frac{\mu_o I^2 s}{D} \quad [\text{independent of } \mu]$$

36. $\overline{F} = \mu_o(\overline{M}\cdot\nabla)\overline{H}$

$$= (\mu - \mu_o)(\overline{H}\cdot\nabla)\overline{H}$$

$$= \nabla(\tfrac{1}{2}(\mu - \mu_o)|\overline{H}|^2)$$

a) $\overline{F} = \frac{1}{2}(\mu - \mu_o)\left\{\frac{\partial}{\partial x}[H_x^2 + H_y^2]\overline{i}_x + \frac{\partial}{\partial y}[H_x^2 + H_y^2]\overline{i}_y\right\}$

$$f_x = \int_{x=-\infty}^{x_o} \int_{y=0}^{s} F_x D dx dy$$

$$= \frac{1}{2}(\mu - \mu_o) D \int_{y=0}^{s} [H_x^2 + H_y^2] \Big|_{x=-\infty}^{x_o} dy$$

$$= \frac{1}{2}(\mu - \mu_o) D \int_{y=0}^{s} H_o^2 dy = \frac{1}{2}(\mu - \mu_o) H_o^2 Ds$$

b) $F_x = \mu_o M_y \dfrac{\partial H_x}{\partial y}$

$$f_x = \mu_o M_y D \int_{y=0}^{s} H_x \Big|_{x=-\infty}^{0} dy$$

$$= \mu_o M_y D[H_o - H_x(-\infty)]s$$

$$= \mu_o M_o D[H_o + M_o]s$$

CHAPTER 6
ELECTROMAGNETIC INDUCTION

Section 6.1

1.

a) $$H_\phi = \frac{I}{2\pi r'} = \frac{I}{2\pi[D + r\cos\phi]}$$

$$\Phi = \int_{r=0}^{a}\int_{\phi=0}^{2\pi} \mu_o H_\phi \, r dr d\phi$$

$$= \frac{\mu_o I}{2\pi}\int_{r=0}^{a}\int_{\phi=0}^{2\pi} \frac{r dr d\phi}{[D + r\cos\phi]}$$

$$= \frac{\mu_o I}{\pi}\int_{r=0}^{a} \frac{2r}{\sqrt{D^2 - r^2}} \tan^{-1}\left\{\frac{\sqrt{D^2 - r^2}\tan\frac{\phi}{2}}{D + r}\right\}\Bigg|_{\phi=0}^{\pi} dr$$

$$= \mu_o I \int_{r=0}^{a} \frac{r}{\sqrt{D^2 - r^2}} dr$$

$$= -\mu_o I \sqrt{D^2 - r^2}\Big|_{r=0}^{a}$$

$$= -\mu_o I\left[\sqrt{D^2 - a^2} - D\right]$$

$$M = \frac{\Phi}{I} = \mu_o\left[D - \sqrt{D^2 - a^2}\right]$$

$$R = \frac{2\pi a}{\sigma A}$$

b) $$-iR = -M\frac{dI}{dt} + L\frac{di}{dt} \rightarrow L\frac{di}{dt} + iR = M\frac{dI}{dt} \longrightarrow i = \frac{MI}{L} e^{-t/\tau} \;;\; \tau = \frac{L}{R}$$

c) $$i(t) = -\frac{MI}{L} e^{-(t - T)/\tau}$$

d) $R = 0 \longrightarrow$ Short circuit: $\frac{d}{dt}[M(D)I - Li] = 0 \longrightarrow M(D)I - Li = M(D_o)I$

$$M(D) = \mu_o\left[D - \sqrt{D^2 - a^2}\right] \;;\; D = D_o + v_r t$$

$$\text{Open circuit:}\quad v_{oc} = \frac{d\Phi}{dt} = -I\frac{dM}{dt} = -I\frac{dM}{dD}\frac{dD}{dt} = -\mu_o I\left[1 - \frac{D}{\sqrt{D^2 - a^2}}\right]v_r$$

e) $$df_r = -\frac{iad\phi\ \mu_o I \cos\phi}{2\pi[D + a\cos\phi]}$$

$$f_r = -\frac{\mu_o Iia}{2\pi}\int_{\phi=0}^{\pi}\frac{2\cos\phi d\phi}{[D + a\cos\phi]}$$

$$= -\frac{\mu_o Iia}{\pi}\left\{-\frac{1}{a}\sin^{-1}[\cos\phi] + \frac{D}{a\sqrt{D^2 - a^2}}\sin^{-1}\left[\frac{a + D\cos\phi}{D + a\cos\phi}\right]\right\}\Bigg|_{\phi=0}^{\pi}$$

$$= \frac{\mu_o Ii}{\pi}\left\{-\frac{\pi}{2} - \frac{\pi}{2} - \frac{D}{\sqrt{D^2 - a^2}}\left[-\frac{\pi}{2} - \frac{\pi}{2}\right]\right\}$$

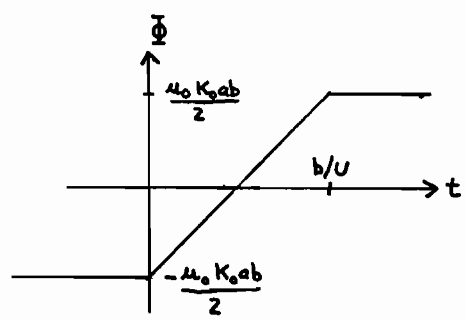
$$= \mu_o Ii\left\{\frac{D}{\sqrt{D^2 - a^2}} - 1\right\}$$

2.

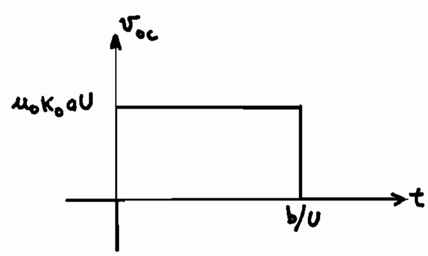
a) $$H_z = \begin{cases} -\dfrac{K_o}{2} & x > 0 \\[2ex] +\dfrac{K_o}{2} & x < 0 \end{cases}$$

$$\Phi = \begin{cases} -\dfrac{\mu_o K_o ab}{2} & t \le 0 \\[2ex] +\dfrac{\mu_o K_o ab}{2} & t \ge \dfrac{b}{U} \end{cases}$$

$$\left\{-\frac{\mu_o K_o a(b - 2Ut)}{2} \qquad 0 \le t \le \frac{b}{U}\right.$$



$$v_{oc} = \frac{d\Phi}{dt}\quad\begin{cases} 0 & t < 0\ ,\ t > \dfrac{b}{U} \\[2ex] \mu_o K_o aU & 0 < t < \dfrac{b}{U} \end{cases}$$



b) $$L\frac{di}{dt} + iR = -\frac{d\Phi}{dt}$$

$t < 0\ ,\ i = 0$

$$0 < t < \frac{b}{U};\ L\frac{di}{dt} + iR = -\mu_o K_o aU \rightarrow i = -\frac{\mu_o K_o aU}{R}\left(1 - e^{-t/\tau}\right)\ ;\ \tau = \frac{L}{R}$$

$$t > \frac{b}{U}\qquad L\frac{di}{dt} + iR = 0 \rightarrow i = -\frac{\mu_o K_o aU}{R}\left(1 - e^{-\frac{b}{U\tau}}\right)e^{-t/\tau}$$

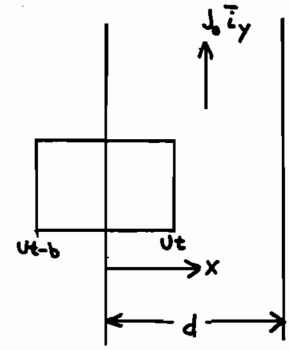
c)
$$H_z = \begin{cases} -\dfrac{J_o d}{2} & x > d \\ \dfrac{J_o d}{2} & x < 0 \\ \dfrac{J_o d}{2}\left(1 - \dfrac{2x}{d}\right) & 0 < x < d \end{cases} \qquad \Phi = \begin{cases} -\dfrac{\mu_o J_o abd}{2} & t < 0 \\ \dfrac{\mu_o J_o abd}{2} & t > \dfrac{d + b}{U} \end{cases}$$

$d > b$, $0 < t < b/U$

$$\Phi = -\frac{\mu_o J_o da(b - Ut)}{2} - \frac{\mu_o J_o da}{2}\int_{x=0}^{Ut}\left(1 - \frac{2x}{d}\right)dx$$

$$= -\frac{\mu_o J_o da}{2}\left[b - Ut - \frac{d\left(1 - \frac{2x}{d}\right)^2}{4}\Bigg|_{x=0}^{Ut}\right]$$

$$= -\frac{\mu_o J_o ad}{2}\left[b - \frac{(Ut)^2}{d}\right]$$

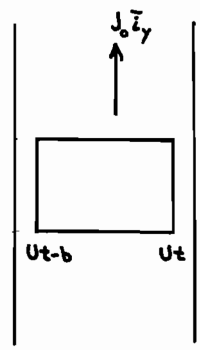


$\frac{b}{U} < t < \frac{d}{U}$

$$\Phi = -\frac{\mu_o J_o da}{2}\int_{Ut-b}^{Ut}\left(1 - \frac{2x}{d}\right)dx$$

$$= \frac{\mu_o J_o d^2 a}{8}\left(1 - \frac{2x}{d}\right)^2\Bigg|_{Ut-b}^{Ut}$$

$$= -\frac{\mu_o J_o dab}{2}\left[\frac{b}{d} + 1 - \frac{2Ut}{d}\right]$$

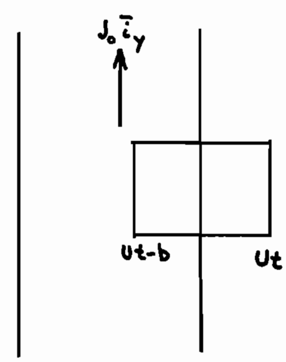


$\frac{d}{U} < t < \frac{d + b}{U}$

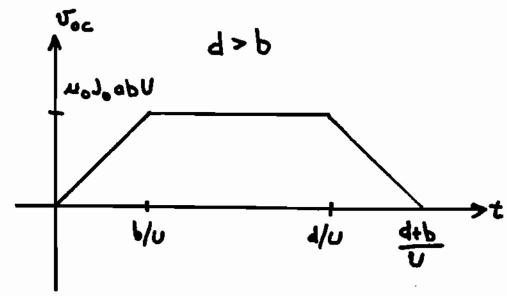
$$\Phi = \frac{\mu_o J_o ad}{2}(Ut - d) - \frac{\mu_o J_o ad}{2}\int_{Ut-b}^{d}\left(1 - \frac{2x}{d}\right)dx$$

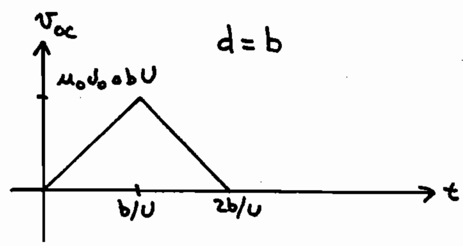
$$= \frac{\mu_o J_o ad}{2}\left[Ut - d + \frac{d}{4}\left(1 - \frac{2x}{d}\right)^2\Bigg|_{x=Ut-b}^{d}\right]$$

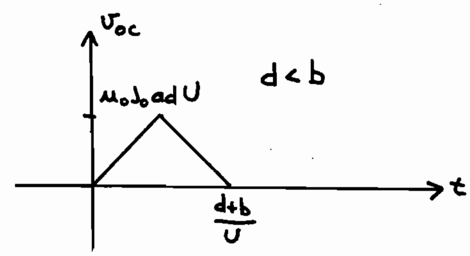
$$= \frac{\mu_o J_o ad}{2}\left[2Ut - (d + b) - \frac{(Ut - b)^2}{d}\right]$$



$$v_{oc} = \frac{d\Phi}{dt} = \begin{cases} 0 & t \leq 0 \\ \mu_o J_o a U^2 t & 0 \leq t \leq \frac{b}{U} \\ \mu_o J_o abU & \frac{b}{U} \leq t \leq \frac{d}{U} \\ \mu_o J_o abU\left[\frac{d}{b} - \frac{(Ut - b)}{b}\right] & \frac{d}{U} \leq t \leq \frac{d + b}{U} \\ 0 & t \geq \frac{d + b}{U} \end{cases}$$







3.

a) $\Phi = B_o\, x\, b \qquad 0 < x < s$

b) $\frac{Ldi}{dt} + iR = \frac{d\Phi}{dt} = B_o b \frac{dx}{dt} = B_o bv \;;\; R = \frac{2(a + b)}{\sigma A}$

c) $m \frac{dv}{dt} = f_x = -iB_o b \rightarrow i = -\frac{m}{B_o b} \frac{dv}{dt}$

$$-\frac{mL}{B_o b} \frac{d^2 v}{dt^2} - \frac{mR}{B_o b} \frac{dv}{dt} = B_o bv$$

Let $\quad \omega_o^2 = \frac{B_o^2 b^2}{mL}, \quad \alpha = \frac{R}{L} \rightarrow \frac{d^2 v}{dt^2} + \alpha \frac{dv}{dt} + \omega_o^2 v = 0$

d) $v = Ve^{st} \rightarrow s^2 + \alpha s + \omega_o^2 = 0 \rightarrow s = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \omega_o^2}$

$$v(t) = \left[V_1 \sin \beta t + V_2 \cos \beta t\right] e^{-\frac{\alpha}{2} t} \;;\; \beta = \sqrt{\omega_o^2 - \left(\frac{\alpha}{2}\right)^2}$$

$$i(t) = -\frac{m}{B_o b} \frac{dv}{dt} = -\frac{m}{B_o b}\left[\left(-\frac{\alpha}{2} V_1 - \beta V_2\right)\sin \beta t + \left(-\frac{\alpha}{2} V_2 + \beta V_1\right)\cos \beta t\right] e^{-\frac{\alpha}{2} t}$$

$$v(t = 0) = v_o = V_2$$

$$i(t = 0) = 0 \rightarrow \beta V_1 = \frac{\alpha}{2} V_2 \rightarrow V_1 = \frac{\alpha}{2\beta} v_o,\; V_2 = v_o$$

$$v(t) = v_o \left[\frac{\alpha}{2\beta} \sin \beta t + \cos \beta t\right] e^{-\frac{\alpha}{2} t}$$

$$i(t) = \frac{m}{B_o b} v_o \left[\frac{\alpha^2}{4\beta} + \beta\right] \sin \beta t \; e^{-\frac{\alpha}{2} t} = \frac{mv_o}{B_o b\beta} \omega_o^2 \sin \beta t \; e^{-\frac{\alpha}{2} t}$$

e) $\sigma \to \infty \to \alpha \to 0 \; , \; \beta = \omega_o$

$$v(t) = v_o \cos \omega_o t$$

$$i(t) = \frac{mv_o \omega_o}{B_o b} \sin \omega_o t$$

$$x(t) = \int v(t)dt = \frac{v_o}{\omega_o} \sin \omega_o t$$

$$x_{max} > s \to \frac{v_o}{\omega_o} > s \to v_o > \frac{B_o bs}{\sqrt{mL}}$$

4.

a) $H_\phi = \dfrac{I}{2\pi r}$

Rectangular cross section toroid

$$\Phi = \mu_o Ns \int_{r=b}^{a} H_\phi dr$$

$$= \frac{\mu_o NsI}{2\pi} \ln \frac{a}{b} \to M = \frac{\Phi}{I} = \frac{\mu_o Ns}{2\pi} \ln \frac{a}{b}$$

Circular cross section toroid

$$\Phi = \frac{\mu_o IN}{2\pi} \int_{r=0}^{a} \int_{\phi=0}^{2} \frac{2rdrd\phi}{R + r \cos \phi}$$

$$= \frac{\mu_o NI}{2\pi} \int_{r=0}^{a} rdr \frac{4}{\sqrt{R^2 - r^2}} \tan^{-1}\left\{\frac{\sqrt{R^2 - r^2} \tan \frac{\phi}{2}}{R + r}\right\}\Bigg|_{\phi=0}^{\pi}$$

$$= \mu_o NI \int_{r=0}^{a} \frac{r}{\sqrt{R^2 - r^2}} dr$$

$$= -\mu_o NI \sqrt{R^2 - r^2}\Big|_{r=0}^{a}$$

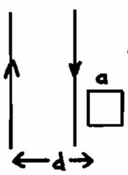
$$= \mu_o NI \left[R - \sqrt{R^2 - a^2}\right] \to M = \frac{\Phi}{I} = \mu_o N\left[R - \sqrt{R^2 - a^2}\right]$$

b) $d+a<D$

$$\Phi = \frac{\mu_o Ib}{2\pi}\left\{\int_d^{d+a}\frac{dr}{r} + \int_{D-d-a}^{D-d}\frac{dr}{r}\right\} = \frac{\mu_o Ib}{2\pi}\ \ell n\left[\left(\frac{d+a}{d}\right)\left(\frac{D-d}{D-d-a}\right)\right]\ ;\ M = \frac{\Phi}{I} = \frac{\mu_o b}{2\pi}\ \ell n\left[\frac{(d+a)(D-d)}{d(D-d-a)}\right]$$

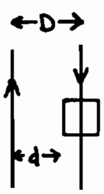
$d>D$

$$\Phi = \frac{\mu_o Ib}{2\pi}\left\{\int_d^{d+a}\frac{dr}{r} - \int_{d-D}^{d+a-D}\frac{dr}{r}\right\} = \frac{\mu_o Ib}{2\pi}\ \ell n\left[\left(\frac{d+a}{d}\right)\left(\frac{d-D}{d+a-D}\right)\right]\ ;\ M = \frac{\Phi}{I} = \frac{\mu_o b}{2\pi}\ \ell n\left[\left(\frac{d+a}{d}\right)\left(\frac{d-D}{d+a-D}\right)\right]$$



$d+a > D$

$$\Phi = \frac{\mu_o Ib}{2\pi}\left\{\int_d^{d+a}\frac{dr}{r} + \int_0^{D-d}\frac{dr}{r} - \int_0^{d+a-D}\frac{dr}{r}\right\}$$



$$= \frac{\mu_o Ib}{2\pi}\ \ell n\left[\left(\frac{d+a}{d}\right)\left(\frac{D-d}{d+a-D}\right)\right]\ ;\ M = \frac{\Phi}{I} = \frac{\mu_o b}{2\pi}\left\{\ell n\left[\left(\frac{d+a}{d}\right)\left(\frac{D-d}{d+a-D}\right)\right]\right\}$$

5.

a) $\bar{A} = \dfrac{\mu_o IdS}{4\pi r^2}\sin\theta\ \bar{i}_\phi\ ,\ \sin\theta = \dfrac{a}{\sqrt{a^2 + D^2}}\ ,\ r = \sqrt{a^2 + D^2}$

b) $A_\phi = \dfrac{\mu_o ma}{4\pi\left(a^2 + D^2\right)^{3/2}} \rightarrow \Phi = \oint_L A_\phi a d\phi = \dfrac{\mu_o ma^2}{2\left(a^2 + D^2\right)^{3/2}}$

c) $M = \dfrac{\Phi}{I} = \dfrac{\mu_o a^2 dS}{2\left(a^2 + D^2\right)^{3/2}}$

d) $B_z(z = -a) = \dfrac{\mu_o I_2 a^2}{2\left(D^2 + a^2\right)^{3/2}}$

e) $\Phi = B_z(z = -a)dS = \dfrac{\mu_o I_2 dSa^2}{2\left(a^2 + D^2\right)^{3/2}}$

f) $M = \dfrac{\Phi}{I_2} = \dfrac{\mu_o dSa^2}{2\left(a^2 + D^2\right)^{3/2}}$ [agrees with (c)]

6.

$$H_z = \begin{cases} -\dfrac{K(t)}{2} & x > 0 \\ +\dfrac{K(t)}{2} & x < 0 \end{cases} \rightarrow \Phi = \frac{\mu_o K_o}{2} s(b - a) = \Phi_o$$

$$L\frac{di}{dt} + iR = \frac{d\Phi}{dt} \;; \quad R = \frac{2(s + a + b)}{\sigma A}$$

a) $i(t) = \frac{\Phi_o}{L} e^{-t/\tau} \;;\; \tau = \frac{L}{R}$, (b) $i(t) = -\frac{\Phi_o}{L} e^{-(t-T)/\tau}$

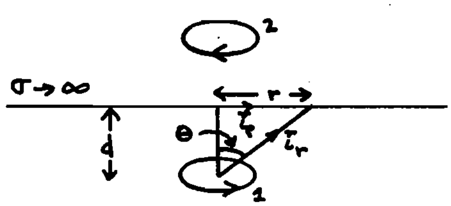
c) $\Phi = \frac{\mu_o K_o}{2} s(b - a) \cos \omega t = \text{Re}\, \Phi_o e^{j\omega t}$

$$i(t) = \text{Re}\hat{I}e^{j\omega t} \rightarrow [Lj\omega + R]\hat{I} = j\omega\Phi_o \rightarrow \hat{I} = \frac{j\omega\Phi_o}{[Lj\omega + R]} = \frac{\Phi_o j\omega}{\left[R^2 + (L\omega)^2\right]^{1/2}} e^{-j\phi}$$

$$\phi = \tan^{-1}\frac{L\omega}{R}$$

$$i(t) = \frac{-\omega\Phi_o}{\left[R^2 + (L\omega)^2\right]^{1/2}} \sin(\omega t - \phi)$$

7. a)



$$\overline{B}_1 = \frac{\mu_o IdS}{4\pi r^3}\left[2\cos\theta\overline{i}_r + \sin\theta\overline{i}_\theta\right] \;;\; \overline{i}_r\cdot\overline{i}_\rho = \sin\theta = \frac{r}{\sqrt{r^2 + d^2}} \;,\; \overline{i}_\phi\cdot\overline{i}_r =$$

$$= \cos\theta = \frac{d}{\sqrt{r^2 + d^2}}$$

$$B_{r1} = \overline{B}_1\cdot\overline{i}_r = \frac{\mu_o IdS}{4\pi r^3}[3\cos\theta\sin\theta]$$

$$B_r = B_{r1} + B_{r2} = \frac{3\mu_o IdS}{2\pi\left(r^2 + d^2\right)^{3/2}}\cos\theta\sin\theta = \frac{3\mu_o IdS\, dr}{2\pi\left(r^2 + d^2\right)^{5/2}}$$

b) $K_\phi = -H_r(z = d) = -\frac{3\, IdS\, rd}{2\pi\left(r^2 + d^2\right)^{5/2}}$

c) $F_z = -\frac{1}{2}\mu_o K_\phi H_r(z = d)$

$= \frac{1}{2}\mu_o K_\phi^2$

$$f_z = \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} F_z \, r dr d\phi$$

$$= \frac{\mu_o 9(IdS)^2 d^2}{4\pi} \int_{r=0}^{\infty} \frac{r^3 dr}{\left(r^2 + d^2\right)^5}$$

$$= -\frac{9\mu_o d^2 (IdS)^2}{4\pi} \left[\frac{r^2 + d^2/4}{6\left(r^2 + d^2\right)^4}\right]\Bigg|_{r=0}^{\infty}$$

$$= \frac{3\mu_o (IdS)^2}{32\pi d^4}$$

d) $$\frac{3\mu_o (IdS)^2}{32\pi d^4} = Mg \rightarrow I^2 = \frac{32\pi Mgd^4}{3\mu_o (dS)^2} = \frac{32\pi\left(10^{-3}\right)(9.8)\left(10^{-8}\right)}{3(4\pi) \times 10^{-7}\left(\pi 10^{-6}\right)^2} = 2.65 \times 10^8$$

$$I = 1.63 \times 10^4 \text{ amperes}$$

8.

a) $H_z = K(t)$

b) $$\oint \overline{E} \cdot d\ell = -\frac{d}{dt} \int_S \overline{B} \cdot \overline{dS}$$

Take contour moving with thin block

$$\int_{y=0}^{s} E'_y dy = \int_{y=0}^{s} \frac{K(t)}{\sigma\delta} dy = \frac{K(t)s}{\sigma\delta} = -\frac{d}{dt}\left[\mu_o s x K(t)\right] = -\mu_o s\left[K(t)\frac{dx}{dt} + x\frac{dK(t)}{dt}\right]$$

$$\frac{dx}{dt} = -V \rightarrow x = x_o - Vt$$

$$\frac{dK}{dt} - \frac{KV}{x_o - Vt}\left[1 - \frac{1}{R_m}\right] = 0 \; ; \; R_m = \sigma\mu_o V\delta$$

$$\frac{1}{K} dK = \frac{dt}{x_o - Vt} V\left(1 - \frac{1}{R_m}\right) \rightarrow \ell n K = -\left(1 - \frac{1}{R_m}\right) \ell n\left(x_o - Vt\right)$$

$$K\left(x_o - Vt\right)^{\left(1 - \frac{1}{R_m}\right)} = \text{constant} = K_o x_o^{\left(1 - \frac{1}{R_m}\right)}$$

$$K(t) = K_o\left(\frac{x_o}{x_o - Vt}\right)^{\left(1 - \frac{1}{R_m}\right)}$$

c) $K(t) = K_o$ if $R_m = \sigma\mu_o V\delta = 1$

d) For $R_m<1$, $K(t)$ decreases with time

For $R_m>1$, $K(t)$ increases with time.

9. $$\oint \overline{E} \cdot \overline{dl} = -\frac{d}{dt}\int_S \overline{B}\cdot\overline{dS}$$

a) $\Phi = \pi r^2 B(t)$, $R = \dfrac{2\pi r}{\sigma d r d}$

$$\oint \overline{E}\cdot\overline{dl} = iR = \frac{d\Phi}{dt} = \pi r^2 \frac{dB}{dt} \rightarrow i = \frac{r\sigma d}{2}\frac{dB}{dt}\, dr$$

b) $$dP = i^2 R = \left(\frac{r\sigma d}{2}\frac{dB}{dt}\right)^2 dr^2 \left(\frac{2\pi r}{\sigma d r d}\right) = \frac{\pi\sigma d}{2}\left(\frac{dB}{dt}\right)^2 r^3\, dr$$

c) $$P = \frac{\pi\sigma d}{2}\left(\frac{dB}{dt}\right)^2 \int_{r=o}^{a} r^3 dr$$

$$= \frac{\pi\sigma d}{2}\left(\frac{dB}{dt}\right)^2 \frac{a^4}{4} = \frac{\pi c d a^4}{8}\left(\frac{dB}{dt}\right)^2$$

d) $\pi a^2 = N\pi(a')^2 \rightarrow a' = \dfrac{a}{\sqrt{N}}$

e) $$P = \frac{\pi\sigma d}{8}\left(\frac{dB}{dt}\right)^2 \frac{a^4}{N^2}\, N = \frac{1}{N}\frac{\pi\sigma d a^4}{8}\left(\frac{dB}{dt}\right)^2$$

<u>Section 6.2</u>

10. $H_\phi = \dfrac{Ni}{2\pi[b+r\cos\theta]}$

$$\Phi = \mu_o \int_{\theta=o}^{2\pi}\int_{r=o}^{a} H_\phi\, r dr d\theta$$

$$= \frac{\mu o N i}{2\pi}\int_{\theta=o}^{\pi}\int_{r=o}^{a} \frac{2 r dr d\theta}{b+r\cos\theta}$$

$$= \frac{\mu_o Ni}{\pi} \int_{r=o}^{a} \frac{2rdr}{\sqrt{b^2-r^2}} \tan^{-1} \frac{\sqrt{b^2-r^2}\tan\frac{\theta}{2}}{b+r} \Bigg|_{\theta=o}^{\pi} dr$$

$$= \mu_o Ni \int_{r=o}^{a} \frac{r}{\sqrt{b^2-r^2}} dr$$

$$= -\mu_o Ni \sqrt{b^2-r^2} \Bigg|_{r=o}^{a} = \mu_o Ni \left[b - \sqrt{b^2-a^2} \right]$$

$$L = \frac{N\Phi}{i} = \mu_o N^2 \left[b - \sqrt{b^2-a^2} \right]$$

11. a) $H_1 = \frac{N_1 i_1}{\ell_1}$, $H_2 = \frac{N_2 i_2}{\ell_2}$

$$\Phi_2 = \mu_o (H_1 + H_2)\pi a_2^2 = \mu_o \pi a_2^2 \left(\frac{N_1}{\ell_1} i_1 + \frac{N_2}{\ell_2} i_2 \right)$$

$$L_2 = \frac{\mu_o \pi a_2^2 N_2^2}{\ell_2}, \quad L_1 = \frac{\mu_o \pi a_1^2 N_1^2}{\ell_1}, \quad M = \frac{\mu_o \pi a_2^2 N_1 N_2}{\ell_2}$$

b) $v_1 = L_1 \frac{di_1}{dt} + \frac{M\,di_2}{dt}$, $v_2 = \frac{M\,di_1}{dt} + \frac{L_2\,di_2}{dt}$

c) $i_1 = i_2 \equiv i$

$$v = v_1 + v_2 = (L_1 + L_2 + 2M) \frac{di}{dt} \rightarrow L_t = L_1 + L_2 + 2M$$

d) $i_1 = - i_2$

$$v = v_1 - v_2 = (L_1 + L_2 - 2M) \frac{di_1}{dt} \rightarrow L_t = L_1 + L_2 - 2M$$

e) $v_1 = v_2 \rightarrow (M - L_2) \frac{di_2}{dt} = (M - L_1) \frac{di_1}{dt} \rightarrow (M - L_2)i_2 = (M - L_1)i_1$

$$i = i_1 + i_2 = i_1 \left[1 + \frac{M-L_1}{M-L_2} \right] \rightarrow i_1 = \frac{M-L_2}{2M-L_1-L_2} i$$

$$v_1 = \frac{di_1}{dt}\left[L_1 + \frac{M(M-L_1)}{M-L_2}\right] = \frac{di}{dt}\left[\frac{M-L_2}{2M-L_1-L_2}\right]\left[-\frac{L_1L_2+M^2}{M-L_2}\right]$$

$$L_t = \frac{M^2-L_1L_2}{2M-L_1-L_2} = \frac{L_1L_2-M^2}{L_1+L_2-2M}$$

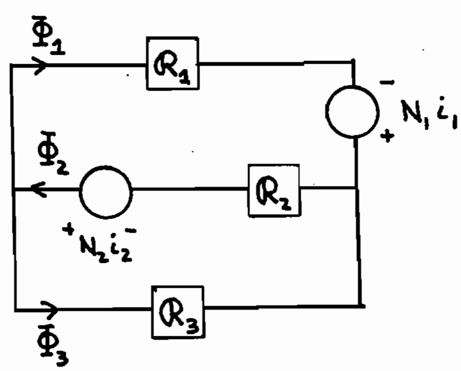
$$v_1 = -v_2 \rightarrow -(M+L_2)\frac{di_2}{dt} = (M+L_1)\frac{di_1}{dt} \rightarrow (M+L_1)i_1 = -(M+L_2)i_2$$

$$i = i_1 - i_2 = i_1\left[1 + \frac{M+L_1}{M+L_2}\right] \rightarrow i_1 = \frac{(M+L_2)i}{2M+L_1+L_2}$$

$$v_1 = \frac{di_1}{dt}\left[L_1 - \frac{M(M+L_1)}{M+L_2}\right] = \frac{di}{dt}\left[\frac{M+L_2}{2M+L_1+L_2}\right]\left[\frac{L_1L_2-M^2}{M+L_2}\right]$$

$$L_t = \frac{L_1L_2-M^2}{L_1+L_2+2M}$$

12. a)



$$R_1 = \frac{s_1}{\mu_1 a_1 D}$$

$$R_2 = \frac{s_2}{\mu_2 a_2 D}$$

$$R_3 = \frac{s_3}{\mu_3 a_3 D}$$

b) $\Phi_2 = \Phi_1 + \Phi_3$

$\Phi_1 R_1 + \Phi_2 R_2 = N_1 i_1 + N_2 i_2$

$\Phi_3 R_3 + \Phi_2 R_2 = N_2 i_2$

$$\rightarrow \Phi_1 = \frac{N_2 i_2 R_3 + N_1 i_1 [R_2 + R_3]}{\left[R_1 + R_3 + \frac{R_1R_3}{R_2}\right] R_2}$$

$$\Phi_2 = \frac{N_2 i_2 [R_1 + R_3] + N_1 i_1 R_3}{R_2\left[R_1 + R_3 + \frac{R_1R_3}{R_2}\right]}$$

$$\Phi_3 = \frac{N_2 i_2 R_1 - N_1 i_1 R_2}{R_2\left[R_1 + R_3 + \frac{R_1R_3}{R_2}\right]}$$

c), d) $\lambda_1 = N_1\Phi_1 = L_1 i_1 + M i_2$; $L_1 = \dfrac{N_1^2[R_3 + R_2]}{R_2\left[R_1 + R_3 + \dfrac{R_1R_3}{R_2}\right]}$

$\lambda_2 = N_2\Phi_2 = M i_1 + L_2 i_2$;

$$L_2 = \frac{N_2^2[R_3 + R_1]}{R_2\left[R_1 + R_3 + \dfrac{R_1R_3}{R_2}\right]}$$

$$M = \frac{N_1N_2R_3}{R_2\left[R_1 + R_3 + \dfrac{R_1R_3}{R_2}\right]}$$

13. a) $H_r = \begin{cases} \dfrac{N_1 I_i}{2g} & o < \Phi < \pi \\ \dfrac{-N_1 I_1}{2g} & \pi < \phi < 2\pi \end{cases}$

b) $\Phi_1 = \mu_o H_r \, \pi a \ell = \dfrac{\mu_o N_1 I_1 a \pi \ell}{2g}$

$$L = \frac{N_1\Phi_1}{I_1} = \frac{\mu_o N_1^2 \pi a \ell}{2g}$$

c)

$$H_r = \begin{cases} \dfrac{[N_1 i_1 - N_2 i_2]}{2g} & o < \phi < \theta \\ \dfrac{[N_1 i_1 + N_2 i_2]}{2g} & \theta < \phi < \pi \\ \dfrac{[-N_1 i_1 + N_2 i_2]}{2g} & \pi < \phi < \pi + \theta \\ \dfrac{-[N_1 i_1 + N_2 i_2]}{2g} & \pi + \theta < \phi < 2\pi \end{cases}$$

d) $\Phi_2 = \dfrac{\mu_o \ell a}{2g} \left\{ (N_1 i_1 + N_2 i_2)(\pi - \theta) + [-N_1 i_1 + N_2 i_2]\theta \right\}$

$$= \frac{\mu_o \ell a}{2g} \left\{ N_1 i_1 (\pi - 2\theta) + N_2 i_2 \pi \right\}$$

$$L_2 = \frac{\mu_o \ell a \pi N_2^{\;2}}{2g} \quad , \quad M = \frac{\mu_o \ell a}{2g} N_1 N_2 \;(\pi - 2\theta)$$

e) $T = 2N_2 I_2 \ell [\mu_o H_r] a$

$$= -\mu_o N_1 N_2 I_1 I_2 \;\frac{a\ell}{2g}$$

14. a) Flux through coil 2 = Φ

Flux through coil 1 = 2Φ

$$\rightarrow \quad v_2 = N_2 \frac{d\Phi}{dt}, \quad v_1 = 2N_1 \frac{d\Phi}{dt} \quad \rightarrow \quad \frac{v_2}{v_1} = \frac{N_2}{2N_1} \; , \; \frac{i_2}{i_1} = \frac{2N_1}{N_2}$$

b) $v_2 = i_2 R_L \rightarrow v_1 \left(\frac{N_2}{2N_1}\right) = \left(\frac{2N_1}{N_2}\right) \; i_1 R_L$

$$\frac{v_1}{i_1} = \left(\frac{2N_1}{N_2}\right)^2 R_L$$

15. a) $\dfrac{v_1}{v_2} = \dfrac{N}{N'} \; , \; \dfrac{i_1}{i_2} = \dfrac{N'}{N}$

b) $v_2 = i_2 R_L \rightarrow \dfrac{N'}{N} \; v_1 = \dfrac{N}{N'} \; i_1 R_L \rightarrow \dfrac{v_1}{i_1} = \left(\dfrac{N}{N'}\right)^2 R_L$

<u>Section 6.3</u>

16. $J_{+x} = qn_+\mu_+E_x \quad , \quad J_{-x} = qn_-\mu_-E_x \qquad\qquad J_x = J_{+x} + J_{-x} = q\;(n_+\mu_+ + n_-\mu_-)E_x$

$J_{+y} = q\mu_+n_+\;(E_y - \mu_+E_xB_z), \; J_{-y} = q\mu_-n_-(E_y + \mu_-E_xB_z)$

$J_y = J_{+y} + J_{-y} = q\;E_y\;(\mu_+n_+ + \mu_-n_-) - q\;E_xB_z\;(\mu_+^{\;2}n_+ - \mu_-^{\;2}n_-)$

a) Open circuit $\rightarrow J_y = 0$

$$E_y = \frac{E_xB_z\left(\mu_+^{\;2}n_+ - \mu_-^{\;2}n_-\right)}{\mu_+n_+ + \mu_-n_-}$$

$$V_{oc} = E_y d = \frac{J_x B_z\, d\left(\mu_+^2 n_+ - \mu_-^2 n_-\right)}{q(\mu_+ n_+ + \mu_- n_-)^2}$$

b) Short circuit $E_y = 0$

$$i_{sc} = J_y \ell s = \frac{-J_x B_z \ell s\left(\mu_+^2 n_+ - \mu_-^2 n_-\right)}{(n_+\mu_+ + n_-\mu_-)^2}$$

17. a) $$B_\phi = \begin{cases} \dfrac{\mu_o I}{2\pi r} & \text{air} \\[2ex] \dfrac{\mu I}{2\pi r} & \text{iron} \end{cases}$$

b) Take contour moving with loop downward $\left(\overline{v} = -V_o \overline{i}_y\right)$

$$= \underbrace{\oint \overline{E}'\cdot\overline{d\ell}}_{\text{EMF}} + \int_{R_1}^{R_2} (\overline{v} \times \overline{B})\cdot\overline{dr} = -\frac{d\Phi}{dt} \quad ; \quad \Phi = \int_S \overline{B}\cdot\overline{dS} = \frac{-I}{2\pi}\left\{[\mu_o y + \mu(L-y)]\int_{R_1}^{R_2}\frac{dr}{r}\right\}$$

$$= \frac{-I\ell n}{2\pi}\,\frac{R_2}{R_1}\,[(\mu_o-\mu)y+\mu L]$$

$$\text{EMF} + \int_{R_1}^{R_2} \frac{V_o \mu I}{2\pi r}\, dr = \frac{I}{2\pi}\,\ell n\,\frac{R_2}{R_1}(\mu-\mu_o)V_o$$

$$\text{EMF} = -\frac{\mu_o V_o I}{2\pi}\,\ell n\,\frac{R_2}{R_1}$$

c) Take stationary contour (velocity of cylinder $\overline{v} = V_o \overline{i}_y$)

$$\underbrace{\oint \overline{E}\cdot\overline{d\ell}}_{\text{EMF}} - \int_{R_1}^{R_2} (\overline{v}\times\overline{B})\cdot\overline{dr} = -\frac{d\Phi}{dt} = \frac{I}{2\pi}\,(\mu-\mu_o)V_o\,\ell n\,\frac{R_2}{R_1}$$

$$\text{EMF} + \frac{V_o \mu I}{2\pi} \ln \frac{R_2}{R_1} = \frac{I}{2\pi} (\mu - \mu_o) V_o \ln \frac{R_2}{R_1}$$

$$\text{EMF} = -\frac{\mu_o I V_o}{2\pi} \ln \frac{R_2}{R_1} \quad \text{(agrees with (b))}$$

d) Closed loop through slot. [Take stationary contour through loop]

$$\underbrace{\oint \overline{E} \cdot \overline{d\ell}}_{\text{EMF}} = -\frac{d\Phi}{dt} = -\frac{IV_o}{2\pi} (\mu - \mu_o) \ln \frac{R_2}{R_1}$$

18. a) $\overline{H} = o, \ \overline{B} = \mu_o M_o \overline{i_z}$

b) $\displaystyle\oint_L \overline{E} \cdot \overline{d\ell} = -\frac{d}{dt} \int_S \overline{B} \cdot \overline{dS}$ (Take stationary contour)

$$\overline{E}' = \overline{E} + \overline{v} \times \overline{B}, \quad \overline{J} = \sigma \overline{E}' = o \rightarrow \overline{E} = -\overline{v} \times \overline{B} \quad \text{in magnet}$$

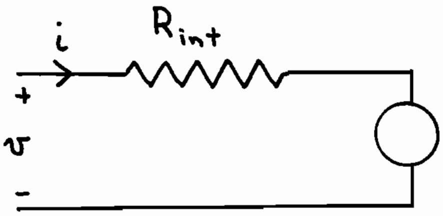
$$\oint_L \overline{E} \cdot \overline{d\ell} = 0 = -v_{oc} + \int_{r=a}^{b} \omega r \overline{i}_\phi \times B_z \overline{i}_z \cdot \overline{dr} = -v_{oc} + \frac{\omega B_z}{2} (b^2 - a^2)$$

$$v_{oc} = \frac{\omega B_z}{2} (b^2 - a^2)$$

c) $\overline{E}' = \dfrac{i}{2\pi r \ell \sigma} \overline{i_r}$

$$\oint \overline{E} \cdot \overline{d\ell} = 0 = -v + \int_{r=a}^{b} \frac{i}{2\pi r \ell \sigma} dr + \int_{r=a}^{b} \omega r \overline{i}_\phi \times B_z \overline{i}_z \cdot \overline{dr}$$

$$v = iR_{int} + \frac{\omega B_z}{2} (b^2 - a^2) \ ; \quad R_{int} = \frac{\ln \frac{b}{a}}{2\pi \sigma \ell}$$



$$+\ \frac{\omega B_z}{2}(b^2-a^2)\ -$$

d) $$v = o \rightarrow i = -\frac{\omega B_z(b^2-a^2)}{2R_{int}}$$

$$J_r = \frac{i}{2\pi r\ell} = -\frac{\omega B_z(b^2-a^2)}{4\pi r\ell\ R_{int}}$$

$$\overline{F} = \overline{J} \times \overline{B} = -\overline{i}_\phi\ \frac{\omega B_z{}^2(b^2-a^2)}{4\pi r\ell\ R_{int}}$$

$$\overline{T} = \int_{r=a}^{b} \overline{r} \times \overline{F}\ 2\pi r\ell dr = -\frac{\omega B_z{}^2(b^2-a^2)\ \overline{i}_z}{2\ R_{int}} \int_{r=a}^{b} r\ dr$$

$$= -\frac{\omega B_z{}^2\left(b^2-a^2\right)^2\ \overline{i}_z}{4\ R_{int}}$$

19. a) $$\oint_L \overline{E}\cdot\overline{d\ell} = -\frac{d}{dt}\int_S \overline{B}\cdot\overline{dS}$$ [Take contour moving with rim]

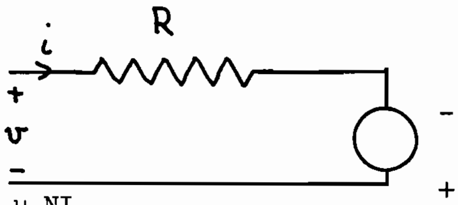
$$-v + \int_a^b \frac{i}{\sigma A}\ dr = -\frac{d}{dt}\left\{(B_o\cos\ \omega t)\frac{(2\pi-\theta)(b^2-a^2)}{2}\right\}\ ;\ o<\theta(t)<2\pi$$

$$v = iR - \frac{(2\pi-\theta)(b^2-a^2)}{2}B_o\ \omega\sin\ \omega t - \frac{B_o(b^2-a^2)\cos\ \omega t\ \omega_o}{2}$$

$$= iR - \frac{B_o(b^2-a^2)}{2}\ [\omega_o\ \cos\ \omega t + \omega(2\pi-\theta)\sin\ \omega t]\ ;\quad R = \frac{b-a}{\sigma A}$$

$$v_{oc} = v(i = o) = -\frac{B_o(b^2-a^2)}{2}\quad [\omega_o\ \cos\ \omega t + \omega(2\pi-\theta)\sin\ \omega t]$$

$$I_{sc} = \frac{B_o(b^2-a^2)}{2R}\ [\omega_o\ \cos\ \omega t + \omega(2\pi-\theta)\ \sin\ \omega t]$$



$$\frac{B_o(b^2-a^2)}{2}[\omega_o \cos \omega t + \omega(2\pi-\theta) \sin \omega t]$$

20. a) $B = \frac{\mu_o N I_o}{s}$

$$J = \frac{i}{sD} = \sigma(-\frac{v}{a} + VB) \;;\; v = -iR_{int} + VBa = iR_L$$

$$i(R_{int} + R_L) = VBa = \frac{\mu_o N I_o Va}{s} \quad ; \; R_{int} = \frac{a}{\sigma s D}$$

$$i = \frac{\mu_o N I_o Va}{s(R_{int}+R_L)} \quad , \; P = i^2 R_L = \left[\frac{\mu_o N Va I_o}{s(R_{int}+R_L)}\right]^2 R_L$$

b) $v_f = \frac{Ldi}{dt}$, $B = \frac{\mu_o Ni}{s}$

$$v_f - v = \frac{Ldi}{dt} + i\,R_{int} - \frac{\mu_o NVa}{s} i = 0$$

$$i = I_o\, e^{-st} \;;\; s = \frac{R_{int} - \frac{\mu_o NVa}{s}}{L}$$

For self-excitation $s < 0 \rightarrow V > \frac{sR_{int}}{\mu_o Na}$

21. $(L_r + L_f)\frac{di_f}{dt} + i_f\,(R_r + R_f - G\omega) + \frac{1}{C}\int i_f dt = 0$

$$\frac{d^2 i_f}{dt^2} + \frac{(R_r + R_f - G\omega)}{(L_r + L_f)}\frac{di_f}{dt} + \frac{1}{(L_r + L_f)C} i_f = 0$$

$$i_f = I\,e^{st}$$

$$s^2 + \frac{(R_r + R_f - G\omega)}{(L_r + L_f)} s + \frac{1}{(L_r + L_f)C} = 0$$

$$s = -\frac{(R_r + R_f - G\omega)}{2(L_r + L_f)} \pm \left[\left[\frac{(R_r + R_f - G\omega)}{2(L_r + L_f)}\right]^2 - \frac{1}{(L_r + L_f)C}\right]^{1/2}$$

a) Self excited if $-G\omega + R_r + R_f < 0 \rightarrow \omega > \frac{(R_r + R_f)}{G}$

b) <u>dc self-excitation</u>

$$\left[\frac{(R_r + R_f - G\,\omega)}{2(L_r + L_f)}\right]^2 - \frac{1}{(L_r + L_f)C} > 0$$

$$C > \frac{4(L_r + L_f)}{[R_r + R_f - G\omega]^2}$$

ac self-excitation

$$\left[\frac{R_r + R_f - G\omega}{2(L_r + L_f)}\right]^2 - \frac{1}{(L_r + L_f)C} < 0 \to C < \frac{4(L_r + L_f)}{[R_r + R_f - G\omega]^2}$$

c) frequency $\omega_o = \left[\frac{1}{(L_r + L_f)C} - \left[\frac{(R_r + R_f - G\omega)}{2(L_r + L_f)}\right]^2\right]^{1/2}$

Section 6.4

22. a) $H_z(t \to \infty) = 0, \quad H_z(t = 0) = \frac{I_o}{D}\left(1 - \frac{x}{d}\right)$

$H_z(x = 0,\ t > 0) = H_z(x = d, t > 0) = 0$

b)

$$H_z(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{d} e^{-\alpha_n t} \quad ; \quad \alpha_n = \frac{1}{\mu\sigma}\left(\frac{n\pi}{d}\right)^2$$

$$H_z(x,\ t = 0) = \frac{I_o}{D}\left(1 - \frac{x}{d}\right) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{d} \to A_n = \frac{2I_o}{n\pi D}$$ [See Eqs.(24)-(26)]

$$H_z(x,t) = \sum_{n=1}^{\infty} \frac{2I_o}{n\pi D} \sin\frac{n\pi x}{d} e^{-\alpha_n t}$$

$$\bar{J}_f = -\bar{i}_y \frac{\partial H_z}{\partial x} = \sum_{n=1}^{\infty} -\frac{2I_o}{dD} \cos\frac{n\pi x}{d} e^{-\alpha_n t}$$

c) $\bar{F} = \mu_o J_y H_z \bar{i}_x = -\mu_o H_z \frac{\partial H_z}{\partial x} \bar{i}_x = -\frac{\partial}{\partial x}\left(\frac{\mu_o}{2} H_z^2\right) \bar{i}_x$

$$f_x = sD \int_{x=o}^{d} F_x dx$$

$$= -s\frac{D\mu_o}{2}\left[H_z^2(x = d) - H_z^2(x = 0)\right] = 0$$

23. a) $H_y(t \to \infty) = H_o = \frac{NI_o}{s}$

b) $H_y(t = 0) = 0$, $H_y(x = 0) = H_y(x = d) = H_o$

c) $$H_y(x,t) = H_o + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{d} e^{-\alpha_n t} \; ; \quad \alpha_n = \frac{1}{\mu\sigma}\left(\frac{n\pi}{d}\right)^2$$

$$H_y(t=0) = 0 \rightarrow \int_{x=0}^{d} H_o \sin\frac{m\pi x}{d}\,dx + \frac{A_m d}{2} = 0$$

$$-\frac{H_o d}{m\pi}\cos\frac{m\pi x}{d}\Big|_{x=0}^{d} + \frac{A_m d}{2} = 0 \rightarrow A_m = \frac{2H_o}{m\pi}(\cos m\pi - 1) = \begin{cases} 0 & m \text{ even} \\ -\dfrac{4H_o}{m\pi} & m \text{ odd} \end{cases}$$

$$H_y(x,t) = H_o + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} -\frac{4H_o}{n\pi}\sin\frac{n\pi x}{d} e^{-\alpha_n t}$$

$$J_z = \frac{\partial H_y}{\partial x} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} -\frac{4H_o}{d}\cos\frac{n\pi x}{d} e^{-\alpha_n t}$$

d) $$F_x = -\mu_o J_z H_y = -\mu_o H_y \frac{\partial H_y}{\partial x} = -\frac{\partial}{\partial x}\left(\tfrac{1}{2}\mu_o H_y^2\right)$$

$$f_x = sD\int_{x=0}^{d} F_x\,dx = -\frac{1}{2}\mu_o sD\, H_y^2\Big|_{x=0}^{d} = 0$$

e) $$H_y = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{d} e^{-\alpha_n(t-T)}$$

$$H_y(t=T) = H_o = \sum_{n-1}^{\infty} A_n \sin\frac{n\pi x}{d} \rightarrow A_n = \frac{4H_o}{n\pi}$$

$$H_y = \sum_{n=1}^{\infty}\frac{4H_o}{n\pi}\sin\frac{n\pi x}{d} e^{-\alpha_n(t-T)} \; ; \quad J_z = \frac{\partial H_y}{\partial x} = \sum_{n=1}^{\infty}\frac{4H_o}{d}\cos\frac{n\pi x}{d} e^{-\alpha_n(t-T)}$$

f) $$H_y(x,t) = \mathrm{Re}\,\hat{H}_y(x)e^{j\omega t}$$

$$\hat{H}_y(x) = A_1 e^{(1+j)x/\delta} + A_2 e^{-(1+j)x/\delta}$$

$$H_y(x=0) = H_o = A_1 + A_2$$

$$H_y(x=d) = H_o = A_1 e^{(1+j)d/\delta} + A_2 e^{-(1+j)d/\delta}$$

$$A_1 = -\frac{H_o\,(1 - e^{-(1+j)d/\delta})}{e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}} \quad , \quad A_2 = \frac{H_o\,(1 - e^{(1+j)d/\delta})}{e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}}$$

$$\hat{H}_y(x) = \frac{H_o}{\left[e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}\right]}\left[- e^{(1+j)x/\delta} + e^{(1+j)(x-d)/\delta} + e^{-(1+j)x/\delta} - e^{-(1+j)(x-d)/\delta}\right]$$

$$\hat{J}_z = \frac{d\hat{H}_y}{dx} = \frac{H_o(1+j)}{\delta[e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}]}\left[- e^{(1+j)x/\delta} + e^{(1+j)(x-d)/\delta} - e^{-(1+j)x/\delta} + e^{-(1+j)(x-d)/\delta}\right]$$

g) From (d)

$$f_x = -\frac{1}{2}\mu_o sD\, H_y^{\,2}\Big|_{x=0}^{d} = 0$$

24. a) $H_z(x,t) = \mathrm{Re}\hat{H}_z(x)e^{j\omega t}$

$$\hat{H}_z(x) = H_o\, e^{-(1+j)x/\delta}; \quad H_o = \frac{I_o}{D}$$

$$\hat{J}_y = -\frac{d\hat{H}_z}{dx} = \frac{(1+j)}{\delta} H_o\, e^{-(1+j)x/\delta}$$

b) $$F_x = \mu_o J_y H_z = -\mu_o H_z \frac{\partial H_z}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{1}{2}\mu_o H_z^{\,2}\right)$$

$$f_x = sD\int_0^\infty F_x dx$$

$$= -sD\int_0^\infty \frac{\partial}{\partial x}\left(\frac{1}{2}\mu_o H_z^{\,2}\right) dx$$

$$= -\frac{1}{2}\mu_o sD\, H_z^{\,2}\Big|_{x=0}^{\infty} = \frac{1}{2}\mu_o sDH_o^{\,2}$$

c) $$\hat{H}_z(x) = A_1\, e^{-(1+j)x/\delta} + A_2 e^{(1+j)x/\delta}$$

$$\hat{H}_x(x=0) = H_o = A_1 + A_2$$

$$\hat{H}_z(x=d) = 0 = A_1 e^{-(1+j)d/\delta} + A_2 e^{(1+j)d/\delta}$$

$$A_1 = -\frac{H_o e^{(1+j)d/\delta}}{e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}} \quad , \quad A_2 = \frac{H_o e^{-(1+j)d/\delta}}{e^{-(1+j)d/\delta} - e^{(1+j)d/\delta}}$$

$$\hat{H}_z(x) = H_o \left[\frac{e^{-(1+j)(x-d)/\delta} - e^{(1+j)(x-d)/\delta}}{e^{(1+j)d/\delta} - e^{-(1+j)d/\delta}} \right]$$

$$\hat{J}_y = -\frac{d\hat{H}_z}{dx} = \frac{H_o(1+j)}{\delta} \left[\frac{e^{-(1+j)(x-d)/\delta} + e^{(1+j)(x-d)/\delta}}{\left[e^{(1+j)d/\delta} - e^{-(1+j)d/\delta} \right]} \right]$$

From (b)

$$f_x = -\frac{1}{2}\mu_o s D H_z^2 \Big|_{x=0}^{d} = \frac{1}{2}\mu_o s D H_o^2 \qquad \text{(unchanged)}$$

25. a) $H_z(t=0) = \begin{cases} -K_o & D<y<D+d \\ o & o<y<D \end{cases}$; $H_z(t \to \infty) = -K_o \quad o<y<D+d$

$J_x(y = 0,t) = 0$

b) $\dfrac{1}{\mu\sigma}\dfrac{\partial^2 H_z}{\partial y^2} = \dfrac{\partial H_z}{\partial t}$; $H_z(y,t) = -K_o + \hat{H}_z(y)e^{-\alpha t}$

$$\frac{d^2\hat{H}_z}{dy^2} + \sigma\mu\alpha\,\hat{H}_z = 0 \to \hat{H}_z = A_1 \sin\sqrt{\sigma\mu\alpha}\,y + A_2 \cos\sqrt{\sigma\mu\alpha}\,y$$

$$\hat{J}_x = \frac{d\hat{H}_z}{dy} = \sqrt{\sigma\mu\alpha}\left[A_1 \cos\sqrt{\sigma\mu\alpha}\,y - A_2 \sin\sqrt{\sigma\mu\alpha}\,y\right]$$

$$\hat{J}_x(y = 0) = \sqrt{\sigma\mu\alpha}\,A_1 = 0 \to A_1 = 0$$

$$\hat{H}_z(y = D) = 0 = A_2 \cos\sqrt{\sigma\mu\alpha}\,D \to \sqrt{\sigma\mu\alpha}\,D = (2n+1)\frac{\pi}{2} \qquad n = 0,1,2\ldots$$

$$\alpha = \frac{1}{\mu\sigma}\left[\frac{(2n+1)\pi}{2D}\right]^2$$

$$H_z(y,t) = -K_o + \sum_{n=o}^{\infty} A_n \cos \frac{(2n+1)\pi y}{2D} e^{-\alpha_n t}$$

$$H_z(y,t=o) = 0 \rightarrow K_o = \sum_{n=o}^{\infty} A_n \cos \left(\frac{(2n+1)\pi y}{2D}\right)$$

$$\int_o^D K_o \cos \frac{(2m+1)\pi y}{2D}\, dy = \frac{2DK_o}{(2m+1)\pi} \sin \frac{(2m+1)\pi y}{2D}\Bigg|_o^D = \frac{2K_oD}{\pi(2m+1)} \sin\left[\frac{(2m+1)\pi}{2}\right]$$

$$= \frac{(-1)^m 2K_oD}{\pi(2m+1)}$$

$$\frac{A_m D}{2} = \frac{(-1)^m 2K_oD}{\pi(2m+1)} \rightarrow A_m = \frac{4K_o(-1)^m}{\pi(2m+1)}$$

$$H_z(y,t) = -K_o + \sum_{n=o}^{\infty} \frac{4K_o (-1)^n}{\pi(2n+1)} \cos\left[\frac{(2n+1)\pi y}{2D}\right] e^{-\alpha_n t}$$

$$J_x(y,t) = \frac{\partial H_z}{\partial y} = \sum_{n=o}^{\infty} -\frac{2K_o}{D} (-1)^n \sin\left[\frac{(2n+1)\pi y}{2D}\right] e^{-\alpha_n t}$$

c) $$H_z(y,t) = \sum_{n=o}^{\infty} A_n \cos\left[\frac{(2n+1)\pi y}{2D}\right] e^{-\alpha_n (t-T)}$$

$$H_z(y,t=T) = -K_o = \sum_{n=o}^{\infty} A_n \cos\left[\frac{(2n+1)\pi y}{2D}\right] \rightarrow A_n = \frac{-4K_o(-1)^n}{\pi(2n+1)}$$

$$H_z(y,t) = \sum_{n=o}^{\infty} \frac{-4K_o (-1)^n}{\pi(2n+1)} \cos\left[\frac{(2n+1)\pi y}{2D}\right] e^{-\alpha_n (t-T)}$$

$$J_x(y,t) = \frac{\partial H_z}{\partial y} = \sum_{n=o}^{\infty} \frac{2K_o}{D} (-1)^n \sin\left[\frac{(2n+1)\pi y}{2D}\right] e^{-\alpha_n (t-T)}$$

d) $$\hat{H}_z(y) = A_1 e^{(1+j)y/\delta} + A_2 e^{-(1+j)y/\delta}$$

$$\hat{J}_x(y) = \frac{d\hat{H}_z}{dy} = \frac{(1+j)}{\delta}\left[A_1 e^{(1+j)y/\delta} - A_2 e^{-(1+j)y/\delta}\right]$$

$$\hat{H}_z(y=D) = K_o = A_1 e^{(1+j)D/\delta} + A_2 e^{-(1+j)D/\delta}$$

$$\hat{J}_x(y=o) = 0 = \frac{(1+j)}{\delta}\left[A_1 - A_2\right] \rightarrow A_1 = A_2 = \frac{K_o}{\left[e^{(1+j)D/\delta} + e^{-(1+j)D/\delta}\right]}$$

$$\hat{H}_z(y) = \frac{K_o}{\left[e^{(1+j)D/\delta} + e^{-(1+j)D/\delta}\right]}\left[e^{(1+j)y/\delta} + e^{-(1+j)y/\delta}\right]$$

$$\hat{J}_x(y=o) = \frac{K_o\ (1+j)}{\delta\left[e^{(1+j)D/\delta} + e^{-(1+j)D/\delta}\right]}\left[e^{(1+j)y/\delta} - e^{-(1+j)y/\delta}\right]$$

26. a) $H_z(x) = A_1 e^{R_m x/\ell} + A_2;\ R_m = \sigma\mu v_o \ell$

$$H_z(x=o) = K_o = A_1 + A_2 \quad \Rightarrow \quad A_1 = \frac{2K_o}{1-e^{R_m}}$$

$$H_z(x=\ell) = -K_o = A_1 e^{R_m} + A_2 \qquad A_2 = \frac{-K_o\left[e^{R_m} + 1\right]}{1-e^{R_m}}$$

$$H_z(x) = \frac{K_o}{1-e^{R_m}}\left[2e^{R_m x/\ell} - (1+e^{R_m})\right]$$

$$J_y(x) = \frac{-\partial H_z}{\partial x} = \frac{-K_o}{1-e^{R_m}}\ \frac{2R_m}{\ell}\ e^{R_m x/\ell}$$

b) $F_x = \mu_o J_y H_z = -\mu_o H_z \frac{\partial H_z}{\partial x} = \frac{-\partial}{\partial x}(\frac{1}{2}\mu_o H_z^{\ 2})$

$$f_x = sD\int_{x=o}^{\ell} F_x dx$$

$$= sD\int_{x=o}^{\ell} \frac{\partial(-\frac{1}{2}\mu_o H_z^2)}{\partial x} dx$$

$$= -\tfrac{1}{2}\mu_o sDH_z^{\ 2}\Big|_{x=o}^{\ell} = 0$$

27. a) $\frac{1}{\mu\sigma}\nabla^2\bar{H} = \frac{\partial\bar{H}}{\partial t} + (\bar{v}\cdot\nabla)\bar{H};\ H_z(x,t) = \mathrm{Re}\ \hat{H}_z(x)\ e^{j\omega t}$

$$\frac{1}{\mu\sigma}\ \frac{d^2\hat{H}_z}{dx^2} = j\omega\hat{H}_z + v_o\frac{d\hat{H}_z}{dx} \rightarrow \frac{d^2\hat{H}_z}{dx^2} - \mu\sigma v_o\frac{d\hat{H}_z}{dx} - j\omega\mu\sigma\hat{H}_z = 0$$

$$\hat{H}_z = Ae^{px} \rightarrow p^2 - \mu\sigma v_o p - j\omega\mu\sigma = 0$$

$$p = \frac{\sigma\mu v_o}{2} \pm \sqrt{\left(\frac{\sigma\mu v_o}{2}\right)^2 + j\omega\mu\sigma}$$

$$= \frac{R_m}{2\ell}\left[1 \pm \sqrt{1 + j\frac{2\ell^2}{R_m{}^2\delta^2}}\right] \quad ; \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$p_1 = \frac{R_m}{2\ell} + \beta$$

$$p_2 = \frac{R_m}{2\ell} - \beta \quad ; \quad \beta = \frac{R_m}{2\ell}\sqrt{1 + \frac{2j\ell^2}{R_m{}^2\delta^2}}$$

$$\hat{H}_z = \left[A_1e^{\beta x} + A_2e^{-\beta x}\right] e^{\frac{R_m x}{2\ell}}$$

$$\hat{H}_z(o) = K_o = A_1 + A_2$$

$$\hat{H}_z(x\rightarrow\infty) = 0 \rightarrow A_1 = 0 \rightarrow A_2 = K_o$$

$$\hat{H}_z(x) = K_o e^{-\beta x} e^{R_m x/2\ell}$$

$$\hat{J}_y(x) = \frac{-d\hat{H}_z}{dx} = -K_o\left(\frac{R_m}{2\ell} - \beta\right) e^{-\beta x} e^{R_m x/2\ell}$$

b) $$F_x = \mu_o J_y H_z = -\mu_o H_z\frac{\partial H_z}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{1}{2}\mu_o H_z{}^2\right]$$

$$f_x = sD\int_{x=o}^{\infty} F_x\,dx = -sD\int_{x=o}^{\infty}\frac{\partial}{\partial x}\left[\frac{1}{2}\mu_o H_z{}^2\right]dx$$

$$= -\frac{1}{2}\mu_o sDH_z{}^2\Big|_o^{\infty} = \frac{1}{2}\mu_o sDK_o{}^2\cos^2\omega t$$

28. a) $$\hat{H}_z(x) = \begin{cases} A_1 e^{kx} + A_2 e^{-kx} & o<x<s \\ B e^{-\gamma(x-s)} & x>s \end{cases} \quad ; H_z(x,z,t) = \mathrm{Re}\,\hat{H}_z(x) e^{j(\omega t - kz)}$$

$$; \gamma^2 = k^2\ (1+jS),\ S = \frac{\mu\sigma}{k^2}\ (\omega - kU)$$

Boundary Conditions:

$$\hat{H}_z\ (x=o) = K_o = A_1 + A_2$$

$$\hat{H}_z\ (x=s_+) = \hat{H}_z\ (x=s_-) = A_1 e^{ks} + A_2 e^{-ks} = B$$

$$\mu_o \hat{H}_x\ (x=s_-) = \mu \hat{H}_x (x=s_+)$$

$$\nabla \cdot \bar{H} = 0 \rightarrow \frac{d\hat{H}_x}{dx} - jk\hat{H}_z = 0 \rightarrow \hat{H}_x = \begin{cases} j\left[A_1 e^{kx} - A_2 e^{-kx}\right] & o<x<s \\ -\frac{B}{\gamma}\ jk e^{-\gamma(x-s)} & x>s \end{cases}$$

$$\mu_o \left[A_1 e^{ks} - A_2 e^{-ks}\right] = -\frac{\mu B k}{\gamma}$$

$$B = \frac{2K_o}{\left[1 - \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{-ks} + \left[1 + \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{ks}}$$

$$A_1 = \frac{K_o\left[1 - \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{-ks}}{\left[1 - \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{-ks} + \left[1 + \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{ks}}, \quad A_2 = \frac{K_o\left[1 + \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{ks}}{\left[1 - \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{-ks} + \left[1 + \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{ks}}$$

$$\hat{\bar{H}}(x) = \begin{cases} \dfrac{K_o\left\{\left[1 - \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{k(x-s)}\ (\bar{i}_z + j\bar{i}_x) + \left[1 + \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{-k(x-s)}\ (\bar{i}_z - j\bar{i}_x)\right\}}{\left[1 - \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{-ks} + \left[1 + \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{ks}} & o<x<s \\[2ex] \dfrac{2K_o\, e^{-\gamma(x-s)}\left[\bar{i}_z - \frac{jk}{\gamma}\bar{i}_x\right]}{\left[1 - \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{-ks} + \left[1 + \frac{\mu}{\mu_o}\frac{k}{\gamma}\right] e^{ks}} & x>s \end{cases}$$

$$J_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \rightarrow \hat{J}_y = -jk\hat{H}_x - \frac{d\hat{H}_z}{dx}$$

$$\hat{J}_y = \begin{cases} o & o<x<s \\ \dfrac{2K_o\left[\gamma^2 - k^2\right]e^{-\gamma(x-s)}}{\gamma\left[\left[1-\frac{\mu}{\mu_o}\frac{k}{\gamma}\right]e^{-ks} + \left[1+\frac{\mu}{\mu_o}\frac{k}{\gamma}\right]e^{ks}\right]} & x>s \end{cases}$$

$$\lim_{s \to o} \begin{cases} \hat{J}_y = \dfrac{K_o e^{-\gamma x}(\gamma^2 - k^2)}{\gamma} & \text{(agrees with (6.4.5.))} \\ \hat{\bar{H}} = K_o e^{-\gamma x}\left[\bar{i}_z - \frac{jk}{\gamma}\bar{i}_x\right] \end{cases}$$

b) $s = o$

$$\hat{H}_z = \begin{cases} A_1 e^{\gamma x} + A_2 e^{-\gamma x} & o<x<d \\ Be^{-k(x-d)} & x>d \end{cases}$$

Boundary Conditions:

$$\hat{H}_z(x=o) = K_o = A_1 + A_2$$

$$\hat{H}_z(x=d_-) = \hat{H}_z(x=d_+) \rightarrow A_1 e^{\gamma d} + A_2 e^{-\gamma d} = B$$

$$\hat{H}_x = \begin{cases} \dfrac{jk}{\gamma}(A_1 e^{\gamma x} - A_2 e^{-\gamma x}) & o<x<d \\ -jBe^{-k(x-d)} & x>d \end{cases}$$

$$\mu H_x(x=d_-) = \mu_o H_x(x=d_+) \Rightarrow \frac{\mu k}{\gamma}\left[A_1 e^{\gamma d} - A_2 e^{-\gamma d}\right] = -\mu_o B$$

$$A_1 = \frac{K_o e^{-\gamma d}\left(1-\frac{\mu_o\gamma}{\mu k}\right)}{\left[\left[1-\frac{\mu_o\gamma}{\mu k}\right]e^{-\gamma d} + \left[1+\frac{\mu_o\gamma}{\mu k}\right]e^{\gamma d}\right]}, \quad A_2 = \frac{K_o e^{\gamma d}\left(1+\frac{\mu_o\gamma}{\mu k}\right)}{\left[\left[1-\frac{\mu_o\gamma}{\mu k}\right]e^{-\gamma d} + \left[1+\frac{\mu_o\gamma}{\mu k}\right]e^{\gamma d}\right]}$$

$$B = \frac{2K_o}{\left[\left[1-\frac{\mu_o\gamma}{\mu k}\right]e^{-\gamma d} + \left[1+\frac{\mu_o\gamma}{\mu k}\right]e^{\gamma d}\right]}$$

$$\hat{H}_z = \begin{cases} \dfrac{K_o\left\{\left(1-\frac{\mu_o\gamma}{\mu k}\right)e^{\gamma(x-d)}\left(\bar{i}_z + \frac{jk}{\gamma}\bar{i}_x\right) + \left(1+\frac{\mu_o\gamma}{\mu k}\right)e^{-\gamma(x-d)}\left(\bar{i}_z - \frac{jk}{\gamma}\bar{i}_x\right)\right\}}{\left[\left[1-\frac{\mu_o\gamma}{\mu k}\right]e^{-\gamma d} + \left[1+\frac{\mu_o\gamma}{\mu k}\right]e^{\gamma d}\right]} & o<x<d \\[2ex] \dfrac{2K_o\left(\bar{i}_z - j\bar{i}_x\right)e^{-k(x-d)}}{\left[\left[1-\frac{\mu_o\gamma}{\mu k}\right]e^{-\gamma d} + \left[1+\frac{\mu_o\gamma}{\gamma k}\right]e^{\gamma d}\right]} & x>d \end{cases}$$

$$J_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \rightarrow \hat{J}_y = -jk\hat{H}_x - \frac{d\hat{H}_z}{dx}$$

$$= \begin{cases} \dfrac{K_o\left\{\left(1-\frac{\mu_o\gamma}{\mu k}\right)e^{\gamma(x-d)} - \left(1+\frac{\mu_o\gamma}{\mu k}\right)e^{-\gamma(x-d)}\right\}\frac{(k^2-\gamma^2)}{\gamma}}{\left[\left[1-\frac{\mu_o\gamma}{\mu k}\right]e^{-\gamma d} + \left[1+\frac{\mu_o\gamma}{\mu k}\right]e^{\gamma d}\right]} & o<x<d \\[2ex] o & x>d \end{cases}$$

Check:

$$\lim_{d\to\infty} \quad \hat{H}_z = K_o\left(\bar{i}_z - \frac{jk}{\gamma}\bar{i}_x\right)e^{-\gamma x}$$

$$\hat{J}_y = \frac{K_o}{\gamma}(\gamma^2-k^2)e^{-\gamma x} \qquad \text{(agrees with (6.4.5))}$$

29. a) $\nabla^2 \bar{H} - \frac{\omega_p^2}{c^2} \bar{H} = 0 \quad ; \; c^2 = \frac{1}{\varepsilon\mu} \quad ; \; \bar{H} = H_y(x)\bar{i}_y$

$$\frac{d^2H_y}{dx^2} - \frac{\omega_p^2}{c^2} H_y = 0 \quad \rightarrow H_y = A_1 \sinh kx + A_2 \cosh kx \; ; \quad k = \frac{\omega_p}{c}$$

$$H_y\,(x=0)=H_o = A_2 \qquad ; \qquad H_o = \frac{NI_o}{s} \cos \omega t$$

$$H_y\,(x=d)=H_o = A_1 \sinh kd + A_2 \cosh kd \quad \rightarrow \quad A_1 = \frac{H_o\,(1-\cosh kd)}{\sinh kd}$$

$$H_y = H_o \left\{ \frac{(1-\cosh kd)}{\sinh kd} \sinh kx + \cosh kx \right\}$$

$$= H_o \left\{ \frac{-2 \sinh^2 \frac{kd}{2} \sinh kx + \cosh kx}{2 \sinh \frac{kd}{2} \cosh \frac{kd}{2}} \right\}$$

$$= \frac{H_o}{\cosh \frac{kd}{2}} \left\{ \cosh kx \cosh \frac{kd}{2} - \sinh \frac{kd}{2} \sinh kx \right\}$$

$$= \frac{H_o}{\cosh \frac{kd}{2}} \cosh k\,(x - \frac{d}{2})$$

$$J_z = \frac{\partial H_y}{\partial x} = \frac{kH_o}{\cosh \frac{kd}{2}} \sinh k\,(x - \frac{d}{2})$$

b) $F_x = -\mu_o J_z H_y = -\mu_o H_y \frac{\partial H_y}{\partial x} = -\frac{\partial}{\partial x}(\frac{1}{2}\mu_o H_y^{\,2})$

$$f_x = sD \int_0^d F_x dx$$

$$= -sD \int_{o}^{d} \frac{\partial}{\partial x}\left(\frac{1}{2}\mu_o H_y{}^2\right) dx$$

$$= -\frac{1}{2}\mu_o sD\, H_y{}^2 \Big|_{o}^{d} = 0$$

Section 6.5

30.

$$H_\phi = \begin{cases} 0 & 0<r<a,\ r>d \\ \dfrac{i\left(r - \dfrac{a^2}{r}\right)}{2\pi\ (b^2 - a^2)} & a<r<b \\ \dfrac{i}{2\pi} & b<r<c \\ \dfrac{-i\left(r - \dfrac{d^2}{r}\right)}{2\pi\ (d^2 - c^2)} & c<r<d \end{cases}$$

$$w = \frac{1}{2}\mu H_\phi{}^2 = \begin{cases} 0 & 0<r<a,\ r>d \\ \frac{1}{2}\mu_1 \left[\dfrac{i}{2\pi(b^2 - a^2)}\right]^2 \left[r^2 + \dfrac{a^4}{r^2} - 2a^2\right] & a<r<b \\ \frac{1}{2}\mu_o \left(\dfrac{i}{2\pi r}\right)^2 & b<r<c \\ \frac{1}{2}\mu_2 \left[\dfrac{i}{2\pi(d^2 - c^2)}\right]^2 \left[r^2 + \dfrac{d^4}{r^2} - 2d^2\right] & c<r<d \end{cases}$$

$$W = 2\pi\ell \int_{r=o}^{d} w r dr$$

$$= 2\pi\ell \left\{ \frac{1}{2}\mu_1 \left[\frac{i}{2\pi(b^2 - a^2)}\right]^2 \int_{r=a}^{b} \left[r^3 + \frac{a^4}{r} - 2a^2 r\right] dr + \frac{1}{2}\mu_0 \left(\frac{i}{2\pi}\right)^2 \int_{r=b}^{c} \frac{dr}{r}\right.$$

$$\left. + \frac{1}{2}\mu_2 \left[\frac{i}{2\pi(d^2 - c^2)}\right]^2 \int_{r=c}^{d} \left[r^3 + \frac{d^4}{r} - 2d^2 r\right] dr \right\}$$

$$= \frac{i^2\ell}{4\pi} \left\{ \frac{\mu_1}{(b^2 - a^2)^2} \left[\frac{b^4 - a^4}{4} + a^4 \ln\frac{b}{a} - a^2(b^2 - a^2)\right] + \mu_0 \ln\frac{c}{b}\right.$$

$$\left. + \frac{\mu_2}{(d^2 - c^2)^2} \left[\frac{d^4 - c^4}{4} + d^4 \ln\frac{d}{c} - d^2(d^2 - c^2)\right]\right\}$$

$$W = \frac{1}{2}Li^2 \Rightarrow L = \frac{2W}{i^2} = \frac{\ell}{2\pi} \left\{ \mu_1 \left[\frac{b^2 - 3a^2}{4(b^2 - a^2)} + \frac{a^4}{(b^2 - a^2)^2} \ln\frac{b}{a}\right] + \mu_0 \ln\frac{c}{b}\right.$$

$$\left. + \mu_2 \left[\frac{c^2 - 3d^2}{4(d^2 - c^2)} + \frac{d^4 \ln\frac{d}{c}}{(d^2 - c^2)^2}\right]\right\}$$

31. $L' C' = \varepsilon\mu \Rightarrow L = L'\ell = \frac{\varepsilon\mu\ell}{C'}$

Adjacent or Enclosed Cylinders

$$C' = \frac{2\pi\varepsilon}{\cosh^{-1}\left[\left|\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right|\right]} \Rightarrow L = \frac{\mu\ell}{2\pi} \cosh^{-1}\left[\left|\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right|\right]$$

Cylinder Plane

$$C' = \frac{2\pi\varepsilon}{\cosh^{-1} \frac{s + R_2}{R_2}} \Rightarrow L = \frac{\mu\ell}{2\pi} \cosh^{-1} \left(\frac{s+R_2}{R_2}\right)$$

32. $$\frac{1}{\mu\sigma} \nabla^2 \bar{H} = \frac{\partial \bar{H}}{\partial t} \quad ; \quad \bar{H} = \mathrm{Re}\, \hat{H}_\phi (r) e^{j\omega t} \,\bar{i}_\phi$$

$$\nabla \times (\hat{H}(\phi)\bar{i}_\phi) = \frac{1}{r} \frac{\partial}{\partial r} (rH_\phi)\bar{i}_z$$

$$\nabla \times (\nabla \times \bar{H}) = -\frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial}{\partial r} (r\hat{H}_\phi)\right]\bar{i}_z = -\nabla^2 \bar{H} = -\mu\sigma j\omega \hat{H}_\phi$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial}{\partial r} (r\hat{H}_\phi)\right] - j\omega\mu\sigma \hat{H}_\phi = 0$$

$$r^2 \frac{d^2 \hat{H}_\phi}{dr^2} + r \frac{d\hat{H}_\phi}{dr} - \hat{H}_\phi (j\omega\mu\sigma r^2 + 1) = 0$$

$$\hat{H}_\phi (r) = A_1 J_1 \left[\frac{r}{\delta} (1-j)\right] \quad ; \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$\hat{H}_\phi (r=a) = \frac{I_o}{2\pi a} \rightarrow A_1 = \frac{I_o}{2\pi a J_1 \left[\frac{a}{\delta} (1-j)\right]}$$

$$\hat{H}_\phi (r) = \frac{I_o}{2\pi a} \frac{J_1 \left[\frac{r}{\delta} (1-j)\right]}{J_1 \left[\frac{a}{\delta} (1-j)\right]}$$

c) $$\hat{J}_z = \frac{1}{r} \frac{d}{dr} (r\hat{H}_\phi) = \frac{d\hat{H}_\phi}{dr} + \frac{\hat{H}_\phi}{r}$$

$$= A_1 \left[\frac{dJ_1}{dr} + \frac{J_1}{r} \right]$$

$$= \frac{I_o}{2\pi a J_1\left[\frac{a}{\delta}(1-j)\right]} \left[J_o \left[\frac{r}{\delta}(1-j)\right]\right] \frac{(1-j)}{\delta}$$

Section 6.6

33. a) $T = \frac{1}{2} i^2 \frac{dL}{d\theta} = - L_1 i_1{}^2 \sin 2\theta = - L_1 I_o{}^2 \cos^2\omega_o t \sin 2\theta$

b) $T = - L_1 I_o{}^2 \left[\frac{1}{2} \sin 2\theta + \frac{1}{4} \left[\sin 2(\omega_o t + \theta) + \sin 2(\theta - \omega_o t) \right]\right]$

$\theta = \omega t + \delta \Rightarrow \langle T \rangle = - \frac{L_1 I_o{}^2}{4} \sin 2\delta$ when $\omega = \pm\omega_o$

c) $\langle T \rangle_{max} = - \frac{1}{4} L_1 I_o{}^2$ when $\delta = \frac{\pi}{4}$

34. a) $\Phi_1 = L_1(\theta) i_1 + M(\theta) i_2$

$\Phi_2 = M(\theta) i_1 + L_2(\theta) i_2$

$v_1 = \frac{d\Phi_1}{dt}, \quad v_2 = \frac{d\Phi_2}{dt}$

$p = v_1 i_1 + v_2 i_2 = i_1 \frac{d\Phi_1}{dt} + i_2 \frac{d\Phi_2}{dt}$

b) $p = i_1 \frac{d}{dt} \left[L_1(\theta) i_1 + M(\theta) i_2 \right] + i_2 \frac{d}{dt} \left[M(\theta) i_1 + L_2(\theta) i_2 \right]$

$$= \frac{d}{dt} \left[\frac{1}{2} L_1(\theta) i_1{}^2 + \frac{1}{2} L_2(\theta) i_2{}^2 + M(\theta) i_1 i_2 \right]$$

$$+\frac{1}{2}\left[i_1^2\frac{dL_1}{d\theta}+i_2^2\frac{dL_2}{d\theta}+2i_1i_2\frac{dM(\theta)}{d\theta}\right]\frac{d\theta}{dt}$$

$$W=\frac{1}{2}L_1(\theta)i_1^2+\frac{1}{2}L_2(\theta)i_2^2+M(\theta)i_1i_2 \qquad \text{energy stored}$$

$$T=\frac{1}{2}i_1^2\frac{dL_1}{d\theta}+\frac{1}{2}i_2^2\frac{dL_2}{d\theta}+i_1i_2\frac{dM}{d\theta} \qquad \text{Torque}$$

c) L_1 = constant, L_2 = constant, $M(\theta) = M_o \sin\theta$

$$T=i_1i_2\frac{dM}{d\theta}=M_oI_1I_2\cos\theta$$

d) $$v_1=L_1\frac{di_1}{dt}+M_oI_2\cos\theta\frac{d\theta}{dt}=-i_1R \quad ; \quad R=\frac{\ell}{\sigma A}$$

e) $$J\frac{d^2\theta}{dt^2}=T=M_oi_1I_2\cos\theta$$

Linearized: $$J\frac{d^2\theta}{dt^2}=M_oI_2i_1$$

$$M_oI_2\frac{d\theta}{dt}=-i_1R-L_1\frac{di_1}{dt}$$

$$\theta=\hat{\theta}e^{st},\ i_1=\hat{I}_1e^{st}$$

$$\begin{aligned}\hat{\theta}\left[Js^2\right]-M_oI_2\hat{I}_1&=0\\ \hat{\theta}\left[sM_oI_2\right]+\hat{I}_1\left[R+L_1s\right]&=0\end{aligned} \quad \rightarrow \quad Js^2(R+L_1s)+(M_oI_2)^2s=0$$

$$s=0,\ s^2+\alpha s+\omega_o^2=0 \quad ; \quad \alpha=\frac{R}{L_1},\ \omega_o^2=\frac{(M_oI_2)^2}{JL_1}$$

$$s=-\frac{\alpha}{2}\pm\sqrt{\left(\frac{\alpha}{2}\right)^2-\omega_o^2}$$

f) $\theta(t) = \left[A_1 e^{j\beta t} + A_2 e^{-j\beta t}\right] e^{-\frac{\alpha}{2} t} + A_3 \; ; \quad \beta = \sqrt{\omega_o^{\,2} - (\frac{\alpha}{2})^2}$

$\theta\ (t \to \infty) = 0 \to A_3 = 0$

$\theta\ (t = o) = \theta_o = A_1 + A_2$

$$\frac{d\theta}{dt}\bigg|_{t=o} = 0 = (j\beta - \frac{\alpha}{2})\ A_1 - (j\beta + \frac{\alpha}{2})\ A_2$$

$$A_1 = \frac{(j\beta + \frac{\alpha}{2})\theta_o}{2j\beta} \qquad A_2 = \frac{(j\beta - \frac{\alpha}{2})\theta_o}{2j\beta}$$

$$\theta(t) = \frac{\theta_o}{2j\beta}\left[(j\beta + \frac{\alpha}{2})e^{j\beta t} + (j\beta - \frac{\alpha}{2})e^{-j\beta t}\right] e^{-\frac{\alpha}{2}t}$$

$$= \theta_o \left[\cos\beta t + \frac{\alpha}{2\beta}\sin\beta t\right] e^{-\frac{\alpha}{2}t}$$

$$i_1(t) = \frac{J}{M_o I_2}\frac{d^2\theta}{dt^2} = \frac{-J\theta_o\omega_o^{\,2}}{MI_2\beta}\left[\beta\cos\beta t - \frac{\alpha}{2}\sin\beta t\right] e^{-\frac{\alpha}{2}t}$$

Solutions: oscillatory if β real $\to \omega_o^2 > \frac{\alpha^2}{4}$

Damped if β imaginary $\to \omega_o^2 < \frac{\alpha^2}{4}$

35. a) $L(x) = \frac{\mu_o x}{2\pi}\ \ln\frac{b}{a}$

b) $f_x = \frac{1}{2}\, i^2\, \frac{dL}{dx} = \frac{\mu_o i^2}{4\pi}\ \ln\frac{b}{a}$

36. <u>Magnetic Circuit</u>

In gaps $H = \frac{NI}{2x} \to \Phi = \frac{\mu_o NI}{2x} sD$

a) $L(x) = \frac{N\Phi}{I} = \frac{\mu_o N^2 sD}{2x} \to$ (b) $f_x = \frac{1}{2} i^2 \frac{dL}{dx} = -\frac{I^2 \mu_o N^2 sD}{4x^2}$

<u>Cylinder-plane</u>

a) $L = \frac{\mu_o D}{2\pi} \cosh^{-1} \frac{x+a}{a}$ (from Prob. 30)

b) $f_x = \frac{1}{2} i^2 \frac{dL}{dx} = \frac{\mu_o D i^2}{4\pi} \frac{1}{a} \frac{1}{[(1+\frac{x}{a})^2 - 1]^{1/2}} = \frac{\mu_o i^2 D}{4\pi[x^2 + 2ax]^{1/2}}$

37. $L(x) = \frac{\ln\frac{b}{a}}{2\pi} [\mu_o(\ell - x) + \mu x]$

$f_x = \frac{1}{2} I^2 \frac{dL}{dx} = \frac{\ln\frac{b}{a} I^2}{4\pi} [\mu - \mu_o] = \rho_m g h \pi (b^2 - a^2)$

$$h = \frac{I^2 \ln\frac{b}{a}(\mu - \mu_o)}{4\pi^2 \rho_m g (b^2 - a^2)}$$

CHAPTER 7

ELECTRODYNAMICS — FIELDS AND WAVES

Section 7.1

1. $\nabla \times \overline{E} = -\mu \dfrac{\partial \overline{H}}{\partial t}$; $\nabla \times \overline{H} = \varepsilon \dfrac{\partial \overline{E}}{\partial t} + \overline{J}_f$; $\nabla \cdot \overline{E} = \dfrac{\rho_f}{\varepsilon}$

a) $\overline{E} = E_o(x\overline{i}_x + y\overline{i}_y)\sin\omega t$

$$\rho_f = \varepsilon \nabla \cdot \overline{E} = 2\varepsilon E_o \sin\omega t$$

$$\nabla \times \overline{E} = 0 \rightarrow \overline{H} = 0 \rightarrow \overline{J}_f = -\varepsilon \frac{\partial \overline{E}}{\partial t} = -\varepsilon E_o(x\overline{i}_x + y\overline{i}_y)\omega\cos\omega t$$

b) $\overline{E} = E_o(y\overline{i}_x - x\overline{i}_y)\cos\omega t$

$$\rho_f = \varepsilon \nabla \cdot \overline{E} = 0$$

$$\nabla \times \overline{E} = \overline{i}_z \left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right] = -2E_o\cos\omega t\,\overline{i}_z = -\mu \frac{\partial \overline{H}}{\partial t} \rightarrow \overline{H} = \frac{2E_o}{\omega\mu}\sin\omega t\,\overline{i}_z$$

$$\overline{J}_f = \nabla \times \overline{H} - \varepsilon \frac{\partial \overline{E}}{\partial t} = +\varepsilon\omega E_o(y\overline{i}_x - x\overline{i}_y)\sin\omega t$$

c) $\overline{E} = \mathrm{Re}\,E_o\, e^{j(\omega t - k_x x - k_z z)}\overline{i}_y$

$$\rho_f = \varepsilon \nabla \cdot \overline{E} = 0$$

$$\nabla \times \overline{E} = -\overline{i}_x \frac{\partial E_y}{\partial z} + \overline{i}_z \frac{\partial E_y}{\partial x} = \mathrm{Re}[-jk_x\overline{i}_z + jk_z\overline{i}_x]E_o e^{j(\omega t - k_x x - k_z z)} = -\mu \frac{\partial \overline{H}}{\partial t}$$

$$\overline{H} = \mathrm{Re}\,\frac{E_o}{\omega\mu}[k_x\overline{i}_z - k_z\overline{i}_x]e^{j(\omega t - k_x x - k_z z)}$$

$$\nabla \times \overline{H} = \overline{i}_y\left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right] = \overline{i}_y \mathrm{Re}\,\frac{jE_o}{\omega\mu}[k_z^2 + k_x^2]\, e^{j(\omega t - k_x x - k_z z)}$$

$$\overline{J}_f = \nabla \times \overline{H} - \varepsilon \frac{\partial \overline{E}}{\partial t} = \overline{i}_y \mathrm{Re}\left[\frac{jE_o}{\omega\mu}(k_x^2 + k_z^2) - j\omega\varepsilon E_o\right]e^{j(\omega t - k_x x - k_z z)}$$

$$\overline{J}_f = 0 \rightarrow k_x^2 + k_z^2 = \omega^2\varepsilon\mu$$

2. $\nabla \times \overline{H} = \overline{J}_f + \varepsilon \frac{\partial \overline{E}}{\partial t} = \sigma \overline{E} + \varepsilon \frac{\partial \overline{E}}{\partial t}$

$$\nabla \cdot \overline{E} = \frac{\rho_f}{\varepsilon}$$

a) $\nabla \cdot \overline{J}_f + \frac{\partial \rho_f}{\partial t} = \frac{\partial \rho_f}{\partial t} + \frac{\rho_f}{\tau} = 0 \rightarrow \rho_f = \rho_o(\overline{r}, t=0)e^{-t/\tau}; \ \tau = \varepsilon/\sigma$

b) $\rho_f = \rho_o e^{-t/\tau}; \ \nabla \cdot \overline{E} = \frac{dE_x}{dx} = \frac{\rho_o}{\varepsilon} e^{-t/\tau} \rightarrow E_x = \frac{\rho_o x}{\varepsilon} e^{-t/\tau} + C(t)$

$$\frac{i(t)}{A} = \sigma E_x + \varepsilon \frac{\partial E_x}{\partial t} = C(t) = (\nabla \times \overline{H})_x$$

Open Circuit: $C(t) = 0 = (\nabla \times \overline{H})_x \rightarrow \overline{H} = 0$

Short Circuit:

$$\int_0^d E_x dx = 0 = \frac{\rho_o d^2}{2\varepsilon} e^{-t/\tau} + C(t)d \rightarrow E_x = \frac{\rho_o}{2\varepsilon}\left(x - \frac{d}{2}\right)e^{-t/\tau}$$

$$(\nabla \times \overline{H})_x = \frac{\partial H_z}{\partial y} = C(t) = -\frac{\rho_o d}{2\varepsilon} e^{-t/\tau} \rightarrow H_z = -\frac{\rho_o dy}{2\varepsilon} e^{-t/\tau}$$

c) $\rho_f = \rho_o e^{-t/\tau}$

$$\nabla \cdot \overline{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) = \frac{\rho_o}{\varepsilon} e^{-t/\tau} \rightarrow E_r = \frac{\rho_o r e^{-t/\tau}}{2\varepsilon} + \frac{C(t)}{r}$$

$$\frac{i(t)}{2\pi r \ell} = \sigma E_r + \varepsilon \frac{\partial E_r}{\partial t} = \frac{C(t)}{r} = (\nabla \times \overline{H})_r$$

Open Circuit: $i(t) = 0 \rightarrow C(t) = 0 \rightarrow \overline{H} = 0$

Short Circuit:

$$\int_a^b E_r dr = 0 = \frac{\rho_o(b^2 - a^2)e^{-t/\tau}}{4\varepsilon} + C(t)\ln\frac{b}{a}$$

$$C(t) = -\frac{\rho_o(b^2 - a^2)e^{-t/\tau}}{4\varepsilon \ln \frac{b}{a}}$$

$$(\nabla \times \overline{H})_r = -\left[\frac{1}{r}\frac{\partial H_\phi}{\partial z}\right] = \frac{C(t)}{r} \rightarrow H_\phi = -C(t)z$$

d) $\overline{H} = 0$ when $i(t) = 0$ [Open circuited terminals]

3. a) $\nabla \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t}$

$$\nabla \cdot \overline{B} = 0 \rightarrow \overline{B} = \nabla \times \overline{A} \rightarrow \nabla \times \left[\overline{E} + \frac{\partial \overline{A}}{\partial t}\right] = 0$$

b) $\overline{E} + \dfrac{\partial \overline{A}}{\partial t} = -\nabla V$

c) $\nabla \times \overline{H} = \dfrac{1}{\mu} \nabla \times (\nabla \times \overline{A}) = \dfrac{1}{\mu}\left[\nabla(\nabla \cdot \overline{A}) - \nabla^2 \overline{A}\right] = \overline{J}_f + \varepsilon \dfrac{\partial \overline{E}}{\partial t}$

$$= \overline{J}_f - \varepsilon\left[\frac{\partial^2 \overline{A}}{\partial t^2} + \nabla \frac{\partial V}{\partial t}\right]$$

$$\nabla^2 \overline{A} - \varepsilon\mu \frac{\partial^2 \overline{A}}{\partial t^2} - \nabla\left[\nabla \cdot \overline{A} + \varepsilon\mu \frac{\partial V}{\partial t}\right] = -\mu \overline{J}_f$$

d) $\nabla \cdot \overline{A} + \varepsilon\mu \dfrac{\partial V}{\partial t} = 0$

e) $\nabla \cdot \overline{E} = -\nabla^2 V - \dfrac{\partial}{\partial t}\left[\nabla \cdot \overline{A}\right] = \dfrac{\rho_f}{\varepsilon}$

From (d) $\quad \nabla^2 V - \varepsilon\mu \dfrac{\partial^2 V}{\partial t^2} = -\dfrac{\rho_f}{\varepsilon}$

f) $r > 0,\ \rho_f = 0$

$$\nabla^2 V - \varepsilon\mu \frac{\partial^2 V}{\partial t^2} = 0; \quad V(r,t) = \mathrm{Re}\hat{V}(r)e^{j\omega t}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{d\hat{V}}{dr}\right) + \frac{\omega^2}{c^2}\hat{V} = 0; \quad c^2 = \frac{1}{\varepsilon\mu}$$

$$\frac{\partial^2}{\partial r^2}(r\hat{V}) + k^2(r\hat{V}) = 0; \quad k = \frac{\omega}{c}$$

$$r\hat{V} = V_1 e^{-jkr} + \cancel{V_2 e^{+jkr}}^{\,0}$$

$$\lim_{r\to 0} \hat{V} = \frac{\hat{Q}}{4\pi\varepsilon r} = \frac{V_1}{r} \rightarrow \hat{V} = \frac{\hat{Q}}{4\pi\varepsilon r} e^{-jkr}$$

<u>Section 7.2</u>

4. a) $\overline{H} \cdot | \nabla \times \overline{E} = -\mu_o \dfrac{\partial}{\partial t}\left[\overline{H} + \overline{M}\right]$

$$-\overline{E}\cdot|(\nabla \times \overline{H}) = \overline{J}_f + \frac{\partial}{\partial t}[\epsilon_o\overline{E} + \overline{P}]$$

$$\nabla\cdot(\overline{E}\times\overline{H}) = \overline{H}\cdot(\nabla\times\overline{E}) - \overline{E}\cdot(\nabla\times\overline{H}) = -\frac{\partial}{\partial t}[\frac{1}{2}\epsilon_o|\overline{E}|^2 + \frac{1}{2}\mu_o|\overline{H}|^2] - \mu_o\overline{H}\cdot\frac{\partial\overline{M}}{\partial t} - \overline{E}\cdot\frac{\partial\overline{P}}{\partial t} - \overline{E}\cdot\overline{J}_f$$

$$\overline{S} = \overline{E}\times\overline{H},\ w = \frac{1}{2}\epsilon_o|\overline{E}|^2 + \frac{1}{2}\mu_o|\overline{H}|^2,\ P_d = \overline{E}\cdot\overline{J}_f,\ P_P = \overline{E}\cdot\frac{\partial\overline{P}}{\partial t},\ P_M = \mu_o\overline{H}\cdot\frac{\partial\overline{M}}{\partial t}$$

$$\nabla\cdot\overline{S} + \frac{\partial w}{\partial t} = -P_d - P_P - P_M$$

b) Energy dissipated per cycle equals area of hysteresis loops

$$w_P = \int P_P dt = \int \overline{E}\cdot d\overline{P} = 4P_S E_C$$

$$w_M = \int P_M dt = \int \mu_o\overline{H}\cdot d\overline{M} = 4\mu_o M_S H_C$$

$$w_T = w_P + w_M = 4[P_S E_C + \mu_o M_S H_C]$$

5. a) $\nabla\cdot(\overline{E}\times\overline{H}) + \frac{\partial}{\partial t}[\frac{1}{2}\epsilon|\overline{E}|^2 + \frac{1}{2}\mu|\overline{H}|^2] = -\overline{E}\cdot\overline{J}_f = \frac{-1}{\omega_p^2\epsilon}\overline{J}_f\cdot\frac{\partial\overline{J}_f}{\partial t}$

$$\nabla\cdot(\overline{E}\times\overline{H}) + \frac{\partial}{\partial t}[\frac{1}{2}\epsilon|\overline{E}|^2 + \frac{1}{2}\mu|\overline{H}|^2 + \frac{1}{2\omega_p^2\epsilon}|\overline{J}_f|^2] = 0$$

$$\overline{S} = \overline{E}\times\overline{H},\ w = \frac{1}{2}\epsilon|\overline{E}|^2 + \frac{1}{2}\mu|\overline{H}|^2 + \frac{1}{2\omega_p^2\epsilon}|\overline{J}_f|^2$$

b) $\overline{J}_f = qn\overline{v} \rightarrow \frac{1}{2\omega_p^2\epsilon}|\overline{J}_f|^2 = \frac{1}{2\omega_p^2\epsilon}q^2n^2|\overline{v}|^2 = \frac{1}{2}nm|\overline{v}|^2$

c) Kinetic energy

d) $\nabla\times\hat{\overline{E}} = -j\omega\mu\hat{\overline{H}}$

$$\nabla\times\hat{\overline{H}} = j\omega\epsilon(1 - \frac{\omega_p^2}{\omega^2})\hat{\overline{E}}$$

$$\nabla\cdot[\frac{1}{2}\hat{\overline{E}}\times\hat{\overline{H}}^*] = \frac{1}{2}[\hat{\overline{H}}^*\cdot(\nabla\times\hat{\overline{E}}) - \hat{\overline{E}}\cdot(\nabla\times\hat{\overline{H}}^*)]$$

$$= \frac{1}{2}\,\hat{H}^*\cdot[-j\omega\mu\hat{H}] + \frac{1}{2}\,\hat{E}\cdot[j\omega\varepsilon(1 - \frac{\omega_P^2}{\omega^2})\hat{E}^*]$$

$$= j\omega[\frac{1}{2}\,\varepsilon|\hat{E}|^2(1 - \frac{\omega_P^2}{\omega^2}) - \frac{1}{2}\,\mu|\hat{H}|^2]$$

e) $\nabla\cdot[\frac{1}{2}\,\hat{E} \times \hat{H}^*] + 2j\omega\langle w\rangle = 0 \rightarrow \langle w\rangle = \frac{-1}{4}\,\varepsilon|\hat{E}|^2(1 - \frac{\omega_P^2}{\omega^2}) + \frac{1}{4}\,\mu|\hat{H}|^2$

6. a) $E_x = \frac{V}{s},\ H_y = \frac{NI}{\ell}$

$S_z = E_x H_y = \frac{NVI}{\ell s}$

b) $w = \frac{1}{2}\,\mu H_y^2 + \frac{1}{2}\,\varepsilon E_x^2 = \frac{1}{2}\,\mu[\frac{NI}{\ell}]^2 + \frac{1}{2}\,\varepsilon[\frac{V}{s}]^2$

c) $\nabla\cdot\overline{S} = 0,\ \frac{\partial w}{\partial t} = 0,\ \nabla\cdot\overline{S} + \frac{\partial w}{\partial t} = 0$

7. $\hat{\overline{E}}(\overline{r}) = \overline{E}_r + j\overline{E}_i$

a) $\hat{\overline{E}}\cdot\hat{\overline{E}} = |\overline{E}_r|^2 - |\overline{E}_i|^2 + 2j\ \overline{E}_r\cdot\overline{E}_i$

$\hat{\overline{E}}\cdot\hat{\overline{E}} = 0$ if $|\overline{E}_r| = |\overline{E}_i|$ and $\overline{E}_r \perp \overline{E}_i$

b) $\hat{\overline{E}}\cdot\hat{\overline{E}}^* = |\overline{E}_r|^2 + |\overline{E}_i|^2$ [Never zero]

c) $\hat{\overline{E}} \times \hat{\overline{E}} = [\overline{E}_r + j\overline{E}_i]\times[\overline{E}_r + j\overline{E}_i] = j[\overline{E}_r \times \overline{E}_i + \overline{E}_i \times \overline{E}_r] = 0$ [Always zero]

d) $\hat{\overline{E}} \times \hat{\overline{E}}^* = [\overline{E}_r + j\overline{E}_i]\times[\overline{E}_r - j\overline{E}_i] = -j[\overline{E}_r \times \overline{E}_i - \overline{E}_i \times \overline{E}_r] = 2j\overline{E}_i \times \overline{E}_r$

[Zero if $\overline{E}_r \parallel \overline{E}_i$]

<u>Section 7.3</u>

8. a) E_x, H_y

$\nabla \times \overline{E} = -\mu\,\frac{\partial\overline{H}}{\partial t} \rightarrow \frac{\partial E_x}{\partial z} = -\mu\,\frac{\partial H_y}{\partial t}$

$\nabla \times \overline{H} = \sigma\overline{E} + \varepsilon\,\frac{\partial\overline{E}}{\partial t} \rightarrow \frac{\partial H_y}{\partial z} = -[\sigma E_x + \varepsilon\,\frac{\partial E_x}{\partial t}]$

b) $\varepsilon\mu \dfrac{\partial^2 E_x}{\partial t^2} + \mu\sigma \dfrac{\partial E_x}{\partial t} - \dfrac{\partial^2 E_x}{\partial z^2} = 0$

c) $\hat{E}_x(z) = \hat{E}\, e^{-jkz} \rightarrow k^2 = \dfrac{\omega^2}{c^2} - j\omega\mu\sigma; \quad c^2 = \dfrac{1}{\varepsilon\mu}$

$\hat{E}_x(z) = \hat{E}_1 e^{-jkz} + \hat{E}_2 e^{+jkz}; \; k = \sqrt{\dfrac{\omega^2}{c^2} + j\omega\mu\sigma}$

d) i) $\dfrac{\sigma}{\omega\varepsilon} << 1, k = \dfrac{\omega}{c}\sqrt{1 - \dfrac{j\sigma}{\omega\varepsilon}} \approx \dfrac{\omega}{c}\left[1 - \dfrac{1}{2}\dfrac{j\sigma}{\omega\varepsilon}\right] \approx \dfrac{\omega}{c} - \dfrac{j}{2}\sigma\sqrt{\dfrac{\mu}{\varepsilon}}$

ii) $\dfrac{\sigma}{\omega\varepsilon} >> 1, \; k \approx \sqrt{-j\omega\mu\,\sigma} = \sqrt{\dfrac{\omega\mu\sigma}{2}}\,(1 - j) = \dfrac{(1 - j)}{\delta}$

e) $\nabla \times \overline{E} = -\mu \dfrac{\partial \overline{H}}{\partial t} \rightarrow \dfrac{\partial E_x}{\partial z} = -\mu \dfrac{\partial H_y}{\partial t}$

$\nabla \times \overline{H} = \overline{J}_f + \varepsilon \dfrac{\partial \overline{E}}{\partial t} \rightarrow \dfrac{\partial^2 H_y}{\partial z \partial t} = -\varepsilon\left[\omega_P^2 E_x + \dfrac{\partial^2 E_x}{\partial t^2}\right]$

$\dfrac{\partial^2 E_x}{\partial t^2} + \omega_P^2 E_x - c^2 \dfrac{\partial^2 E_x}{\partial z^2} = 0; \; E_x(z,t) = \mathrm{Re}\,\hat{E}_x e^{j(\omega t - kz)}$

$k^2 = \dfrac{\omega^2 - \omega_P^2}{c^2} \rightarrow k = \pm\dfrac{\sqrt{\omega^2 - \omega_P^2}}{c}$

f) $H_y = \begin{cases} -\dfrac{K_o}{2}\,\mathrm{Re}\, e^{j(\omega t - kz)} \\ \dfrac{K_o}{2}\,\mathrm{Re}\, e^{j(\omega t + kz)} \end{cases}, \quad E_x = \dfrac{\omega\mu H_y}{\pm k} = \begin{cases} -\dfrac{K_o}{2}\,\mathrm{Re}\,\dfrac{\omega\mu}{k}\, e^{j(\omega t - kz)} \\ -\dfrac{K_o}{2}\,\mathrm{Re}\,\dfrac{\omega\mu}{k}\, e^{j(\omega t + kz)} \end{cases}$

9. a) $\nabla \times \overline{E} = -\mu \dfrac{\partial \overline{H}}{\partial t} \rightarrow \dfrac{\partial E_x}{\partial z} = -\mu \dfrac{\partial H_y}{\partial t}$

$\nabla \times \overline{H} = \overline{J}_f + \varepsilon \dfrac{\partial \overline{E}}{\partial t} \rightarrow -\dfrac{\partial H_y}{\partial z} = J_x + \varepsilon \dfrac{\partial E_x}{\partial t}$

$\varepsilon \dfrac{\partial^2 E_x}{\partial t^2} + \dfrac{\partial J_x}{\partial t} = -\dfrac{\partial^2 H_y}{\partial z \partial t} = \dfrac{1}{\mu}\dfrac{\partial^2 E_x}{\partial z^2}$

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \mu\frac{\partial J_x}{\partial t}\ ; \quad c^2 = \frac{1}{\varepsilon\mu}$$

b) Homogeneous Solution: $E_x = \left[\mathrm{Re}\ \hat{E}_1 e^{-jkz} + \hat{E}_2 e^{jkz}\right] e^{j\omega t};\ k = \frac{\omega}{c}$

Particular Solution: $E_x = \mathrm{Re}\left[\frac{\mu c^2 j}{\omega} J_o e^{j\omega t}\right]$

$$\hat{E}_x(z) = \begin{cases} \hat{E}_1 e^{-jk_o(z-d)} & z > d \quad ; \quad k_o = \omega\sqrt{\varepsilon_o\mu_o} \\ \hat{E}_2 e^{-jkz} + \hat{E}_3 e^{+jkz} + \frac{jJ_o}{\omega\varepsilon} & -d < z < d \quad ; \quad k = \omega\sqrt{\varepsilon\mu} \\ \hat{E}_4 e^{+jk_o(z+d)} & z < -d \end{cases}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu}\frac{\partial E_x}{\partial z} \rightarrow \hat{H}_y(z) = \begin{cases} \frac{\hat{E}_1}{\eta_o} e^{-jk_o(z-d)} & z > d \qquad \eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \\ \frac{1}{\eta}\left[\hat{E}_2 e^{-jkz} - \hat{E}_3 e^{jkz}\right] & -d < z < d \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}} \\ -\frac{\hat{E}_4}{\eta_o} e^{jk_o(z+d)} & z < -d \end{cases}$$

<u>Boundary Conditions</u>:

$$E_x(z=d_+) = E_x(z=d_-) \rightarrow \hat{E}_1 = \hat{E}_2 e^{-jkd} + \hat{E}_3 e^{jkd} + \frac{jJ_o}{\omega\varepsilon}$$

$$H_y(z=d_+) = H_y(z=d_-) \rightarrow \frac{\hat{E}_1}{\eta_o} = \frac{1}{\eta}\left[\hat{E}_2 e^{-jkd} - \hat{E}_3 e^{jkd}\right]$$

$$E_x(z=-d_+) = E_x(z=-d_-) \rightarrow \hat{E}_4 = \hat{E}_2 e^{jkd} + \hat{E}_3 e^{-jkd} + \frac{jJ_o}{\omega\varepsilon}$$

$$H_y(z=-d_+) = H_y(z=-d_-) \rightarrow -\frac{\hat{E}_4}{\eta_o} = \frac{1}{\eta}\left[\hat{E}_2 e^{jkd} - \hat{E}_3 e^{-jkd}\right]$$

Symmetry $\rightarrow \hat{E}_1 = \hat{E}_4,\ \hat{E}_2 = \hat{E}_3$

$$\hat{E}_1 = \hat{E}_4 = \frac{j\eta_o J_o \sin kd}{\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]}\ ; \quad \hat{E}_2 = \hat{E}_3 = -\frac{J_o\eta}{2\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]}$$

$$\hat{E}_x(z) = \begin{cases} \dfrac{j\eta_o J_o \sin kd\, e^{-jk_o(z-d)}}{\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]} & z > d \\[2ex] \dfrac{-J_o\eta\cos kz}{\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]} + \dfrac{jJ_o}{\omega\varepsilon} & -d < z < d \\[2ex] \dfrac{j\eta_o J_o \sin kd e^{+jk_o(z+d)}}{\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]} & z < -d \end{cases}$$

$$\hat{H}_y(z) = \begin{cases} \dfrac{jJ_o \sin kd e^{-jk_o(z-d)}}{\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]} & z > d \\[2ex] \dfrac{jJ_o \sin kz}{\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]} & -d < z < d \\[2ex] \dfrac{-jJ_o \sin kd e^{jk_o(z+d)}}{\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]} & z < -d \end{cases}$$

c) $$P_V = -\frac{1}{2}\,\mathrm{Re}\int_{-d}^{d} \hat{\underline{E}}\cdot\hat{\underline{J}}_f^{*}\,dz$$

$$= \mathrm{Re}\left[\frac{1}{2}\,\frac{J_o^2\eta}{\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]}\int_{-d}^{+d}\cos kz dz + \frac{jJ_o^2}{\omega\varepsilon}\right]$$

$$= \mathrm{Re}\left[\frac{J_o^2\eta \sin kd}{k\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]}\right]$$

$$= \frac{J_o^2\eta_o\eta\sin^2 kd}{k\omega\varepsilon[(\eta_o \sin kd)^2 + (\eta\cos kd)^2]}$$

d) $$P_z(z{=}d) = \frac{1}{2}\,\mathrm{Re}(\hat{E}_x(d)\hat{H}_y^{*}(d))$$

$$= \frac{1}{2}\,\mathrm{Re}\left[\frac{j\eta_o J_o \sin kd}{\omega\varepsilon[\eta_o \sin kd - j\eta\cos kd]}\;\frac{(-jJ_o \sin kd)}{\omega\varepsilon[\eta_o \sin kd + j\eta\cos kd]}\right]$$

$$= \frac{1}{2}\,\frac{\eta_o J_o^2 \sin^2 kd}{(\omega\varepsilon)^2[(\eta_o \sin kd)^2 + (\eta\cos kd)^2]}$$

$$k = \omega\sqrt{\varepsilon\mu} \rightarrow \frac{\eta}{k\omega\varepsilon} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\omega^2\varepsilon\sqrt{\varepsilon\mu}} = \frac{1}{(\omega\varepsilon)^2}$$

$$P_z(z=d) = -P_z(z=-d) = \frac{1}{2} P_V$$

10. a) $$\nabla \times \overline{E} = -\mu(z)\frac{\partial\overline{H}}{\partial t} \rightarrow \frac{\partial E_x}{\partial z} = -\mu(z)\frac{\partial H_y}{\partial t}$$

$$\nabla \times \overline{H} = \varepsilon(z)\frac{\partial\overline{E}}{\partial t} \rightarrow \frac{\partial H_y}{\partial z} = -\varepsilon(z)\frac{\partial E_x}{\partial t}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{\varepsilon(z)}\frac{\partial H_y}{\partial z}\right) = -\frac{\partial^2 E_x}{\partial z\,\partial t} = \mu(z)\frac{\partial^2 H_y}{\partial t^2}$$

$$\frac{\partial^2 H_y}{\partial t^2} - \frac{1}{\mu(z)}\frac{\partial}{\partial z}\left[\frac{1}{\varepsilon(z)}\frac{\partial H_y}{\partial z}\right] = 0$$

$$\frac{\partial}{\partial z}\left(\frac{1}{\mu(z)}\frac{\partial E_x}{\partial z}\right) = -\frac{\partial^2 H_y}{\partial z\,\partial t} = \varepsilon(z)\frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_x}{\partial t^2} - \frac{1}{\varepsilon(z)}\frac{\partial}{\partial z}\left[\frac{1}{\mu(z)}\frac{\partial E_x}{\partial z}\right] = 0$$

b) $\varepsilon(z) = \varepsilon e^{\alpha|z|}$, $\mu(z) = \mu e^{-\alpha|z|}$

$$\frac{\partial^2 E_x}{\partial t^2} - \frac{1}{\varepsilon(z)\mu(z)}\frac{\partial^2 E_x}{\partial z^2} + \frac{1}{\varepsilon(z)\mu^2(z)}\frac{\partial E_x}{\partial z}\frac{d\mu}{dz} = 0$$

$$\frac{\partial^2 E_x}{\partial t^2} - \frac{1}{\varepsilon\mu}\frac{\partial^2 E_x}{\partial z^2} \mp \frac{\alpha}{\varepsilon\mu}\frac{\partial E_x}{\partial z} = 0 \qquad \begin{matrix} z > 0 \\ z < 0 \end{matrix}$$

$$E_x(z,t) = \mathrm{Re}\hat{E}_x(z)e^{j(\omega t - k|z|)}$$

$$-\omega^2 + \frac{k^2}{\varepsilon\mu} + \frac{\alpha jk}{\varepsilon\mu} = 0 \rightarrow k^2 + \alpha jk - \omega^2\varepsilon\mu = 0$$

$$k = -\frac{j\alpha}{2} \pm \sqrt{-\frac{\alpha^2}{4} + \omega^2\varepsilon\mu} = -\frac{j\alpha}{2} \pm \beta; \quad \beta = \sqrt{\omega^2\varepsilon\mu - \frac{\alpha^2}{4}}$$

$$E_x = \begin{cases} E_o e^{j(\omega t - \beta z)} e^{-\frac{\alpha}{2} z} & z > 0 \\ E_o e^{j(\omega t + \beta z)} e^{+\frac{\alpha}{2} z} & z < 0 \end{cases}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu(z)} \frac{\partial E_x}{\partial z} \rightarrow H_y = \begin{cases} \frac{(j\beta + \frac{\alpha}{2})}{j\omega\mu} E_o e^{j(\omega t - \beta z)} e^{+\frac{\alpha}{2} z} & z > 0 \\ \frac{-(j\beta + \frac{\alpha}{2})}{j\omega\mu} E_o e^{j(\omega t + \beta z)} e^{-\frac{\alpha}{2} z} & z < 0 \end{cases}$$

$$H_y(z=0_-) - H_y(z=0_+) = K_o e^{j\omega t} \rightarrow \frac{-2(j\beta + \frac{\alpha}{2})}{j\omega\mu} E_o = K_o \rightarrow E_o = \frac{j\omega\mu K_o}{-2(j\beta + \frac{\alpha}{2})}$$

c) $\omega^2 \varepsilon\mu - \frac{\alpha^2}{4} > 0$ oscillatory

$\omega^2 \varepsilon\mu - \frac{\alpha^2}{4} < 0$ evanescent

11. a) $z' = \gamma(z - vt)$, $t' = \gamma(t - vz/c_o^2)$; $\gamma = \frac{1}{\sqrt{1 - (\frac{v}{c_o})^2}}$

$z_1' = \gamma(z_1 - vt)$, $z_2' = \gamma(z_2 - vt)$

$t_1' = \gamma(t_1 - vz_1/c_o^2)$, $t_2' = \gamma(t_2 - vz_2/c_o^2)$

$$t_1' - t_2' = \frac{\gamma v}{c_o^2}(z_2 - z_1)$$

b) $t_1' = \gamma(t_1 - \frac{vz}{c_o^2})$, $t_2' = \gamma(t_2 - \frac{vz}{c_o^2})$

$$t_1' - t_2' = \gamma(t_1 - t_2)$$

c) $z_2' - z_1' = \gamma(z_2 - z_1) = \gamma L$

12. a) $\overline{u} = u_x\overline{i}_x + u_y\overline{i}_y + u_z\overline{i}_z$; $z' = \gamma(z - vt), t' = \gamma(t - \frac{vz}{c_o^2})$; $\gamma = \frac{1}{\sqrt{1-(\frac{v}{c_o})^2}}$

$$\overline{u}' = \frac{d\overline{r}'}{dt'}$$

$$u'_z = \frac{\Delta z'}{\Delta t'} = \frac{\Delta z - v\Delta t}{\Delta t - \frac{v\Delta z}{c_o^2}} = \frac{\frac{\Delta z}{\Delta t} - v}{1 - \frac{v}{c_o^2}\frac{\Delta z}{\Delta t}} = \frac{u_z - v}{1 - \frac{vu_z}{c_o^2}}$$

$$u'_x = \frac{\Delta x'}{\Delta t'} = \frac{\Delta x}{\gamma(\Delta t - \frac{v\Delta z}{c_o^2})} = \frac{\Delta x/\Delta t}{\gamma(1 - \frac{v}{c_o^2}\frac{\Delta z}{\Delta t})} = \frac{u_x\sqrt{1-(\frac{v}{c_o})^2}}{1 - \frac{vu_z}{c_o^2}}$$

$$u'_y = \frac{u_y\sqrt{1-(\frac{v}{c_o})^2}}{1 - \frac{vu_z}{c_o^2}}$$

b) i) $\overline{u} = c_o\overline{i}_x \rightarrow u'_x = c_o\sqrt{1-(\frac{v}{c_o})^2}$, $u'_y = 0$, $u'_z = -v$

ii) $\overline{u} = c_o\overline{i}_y \rightarrow u'_x = 0$, $u'_y = c_o\sqrt{1-(\frac{v}{c_o})^2}$, $u'_z = -v$

iii) $\overline{u} = c_o\overline{i}_z \rightarrow u'_x = 0$, $u'_y = 0$, $u'_z = \frac{c_o - v}{1 - \frac{v}{c_o}} = c_o$

iv) $\overline{u} = \frac{c_o}{\sqrt{3}}[\overline{i}_x + \overline{i}_y + \overline{i}_z] \rightarrow u'_x = u'_y = \frac{c_o}{\sqrt{3}}\sqrt{1-(\frac{v}{c_o})^2}$, $u'_z = \frac{\frac{c_o}{\sqrt{3}} - v}{1 - \frac{v}{c_o\sqrt{3}}}$

c) $$\sqrt{u'^2_x + u'^2_y + u'^2_z} = \frac{[(u_z - v)^2 + (u_x^2 + u_y^2)(1 - (\frac{v}{c_o})^2)]^{1/2}}{1 - \frac{vu_z}{c_o^2}}$$

$$= \frac{\{u_x^2 + u_y^2 + u_z^2 - (\frac{v}{c_o})^2[u_x^2 + u_y^2 - c_o^2] - 2vu_z\}^{1/2}}{1 - \frac{vu_z}{c_o^2}}$$

If $u_x^2 + u_y^2 + u_z^2 = c_o^2$

$$\sqrt{u_x'^2 + u_y'^2 + u_z'^2} = \frac{\{c_o^2 + (\frac{vu_z}{c_o})^2 - 2vu_z\}^{1/2}}{1 - \frac{vu_z}{c_o}}$$

$$= \frac{c_o - \frac{vu_z}{c_o}}{1 - \frac{vu_z}{c_o^2}} = c_o$$

Section 7.4

13. $\overline{E} = 100e^{j(2\pi \times 10^6 t - 2\pi \times 10^{-2} z)}\overline{i}_x$

a) $\omega = 2\pi \times 10^6$ rad/sec $\rightarrow f = \frac{\omega}{2\pi} = 10^6$ Hz

$k = 2\pi \times 10^{-2}\ m^{-1} \rightarrow \lambda = \frac{2\pi}{k} = 10^2$ meter

$c = \frac{\omega}{k} = f\lambda = 10^8$ meter/sec

b) $c^2 = \frac{1}{\varepsilon\mu_o} \rightarrow \varepsilon = \frac{1}{c^2\mu_o} = \frac{10^{-9}}{4\pi} = \varepsilon_r\varepsilon_o = \frac{\varepsilon_r \times 10^{-9}}{36\pi} \rightarrow \varepsilon_r = 9$

$\eta = \sqrt{\frac{\mu_o}{\varepsilon}} = 40\pi$

$H_y = \frac{E_x}{\eta} = \frac{2.5}{\pi} e^{j(2\pi \times 10^6 t - 2\pi \times 10^{-2} z)}$ amps/meter

c) $\langle S\rangle_z = \frac{1}{2}\frac{|\overline{E}|^2}{\eta} = \frac{10^4}{80\pi} = \frac{125}{\pi}$ watts/meter^2

14. $\overline{E} = \mathrm{Re}[E_{xo}\overline{i}_x + E_{yo}e^{j\phi}\overline{i}_y]e^{j(\omega t - kz)} = E_{xo}\cos(\omega t - kz)\overline{i}_x + E_{yo}\cos(\omega t - kz + \phi)\overline{i}_y$

a) $\overline{H} = \frac{1}{\eta}\mathrm{Re}[E_{xo}\overline{i}_y - E_{yo}e^{j\phi}\overline{i}_x]e^{j(\omega t - kz)}$

$$= \frac{E_{xo}}{\eta}\cos(\omega t - kz)\overline{i}_y - \frac{E_{yo}}{\eta}\cos(\omega t - kz + \phi)\overline{i}_x$$

b) $\overline{S} = \overline{E} \times \overline{H} = [\frac{E_{xo}^2}{\eta} \cos^2(\omega t - kz) + \frac{E_{yo}^2}{\eta} \cos^2(\omega t - kz + \phi)]\overline{i}_z$

$$\langle \overline{S} \rangle = \frac{1}{2} [\frac{E_{xo}^2 + E_{yo}^2}{\eta}] \overline{i}_z$$

15. a) $M \frac{d^2\overline{d}}{dt^2} + \frac{Q^2}{4\pi\varepsilon_o R_o^3} \overline{d} = QE_o \cos\omega t \rightarrow \frac{d^2\overline{d}}{dt^2} + \omega_o^2 \overline{d} = \frac{QE_o}{M} \cos\omega t; \ \omega_o^2 = \frac{Q^2}{M4\pi\varepsilon_o R_o^3}$

$$\overline{d} = \frac{QE_o}{M(\omega_o^2 - \omega^2)} \cos\omega t$$

b) $\overline{P} = NQ\overline{d} = \frac{NQ^2E_o}{M(\omega_o^2 - \omega^2)} \cos\omega t = \frac{\varepsilon_o E_o \omega_p^2}{\omega_o^2 - \omega^2} \cos\omega t; \ \omega_p^2 = \frac{Q^2 N}{M\varepsilon_o}$

$$\overline{D} = \varepsilon_o \overline{E} + \overline{P} \rightarrow \hat{\overline{D}} = \varepsilon_o [1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}] E_o \cos\omega t \rightarrow \varepsilon(\omega) = \varepsilon_o [1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}]$$

c) $k^2 = \omega^2 \varepsilon(\omega) \mu_o = \frac{\omega^2}{c^2} [1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}]; \ c^2 = \frac{1}{\varepsilon_o \mu_o}$

d) Propagation $\rightarrow$ k real $\rightarrow \omega < \sqrt{\omega_o^2 + \omega_p^2}$

Evanescence $\rightarrow$ k imaginary $\rightarrow \omega > \sqrt{\omega_o^2 + \omega_p^2}$

e) $v_p = \frac{\omega}{k} = \frac{c}{[1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}]^{1/2}}$

$$2k \frac{dk}{d\omega} = \frac{1}{c^2} \left[2\omega[1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}] + \frac{\omega^2 \omega_p^2 2\omega}{(\omega_o^2 - \omega^2)^2} \right]$$

$$= \frac{2\omega}{c^2} \left[\frac{(\omega_o^2 - \omega^2)^2 + \omega_p^2(\omega_o^2 - \omega^2) + \omega^2 \omega_p^2}{(\omega_o^2 - \omega^2)^2} \right]$$

$$= \frac{2\omega}{c^2(\omega_o^2 - \omega^2)^2} [(\omega_o^2 - \omega^2)^2 + \omega_p^2 \omega_o^2]$$

$$v_g = \frac{d\omega}{dk} = \frac{kc^2}{\omega} \frac{(\omega_o^2 - \omega^2)^2}{[(\omega_o^2 - \omega^2)^2 + \omega_p^2\omega_o^2]}$$

f) $\nabla \times \hat{\overline{E}} = -j\omega\mu\hat{\overline{H}}$

$$\nabla \times \hat{\overline{H}}^* = -j\omega\varepsilon(\omega)\hat{\overline{E}}^*$$

$$\nabla\cdot[\hat{\overline{E}} \times \hat{\overline{H}}^*] = \hat{\overline{H}}^*\cdot(\nabla \times \hat{\overline{E}}) - \hat{\overline{E}}\cdot(\nabla \times \hat{\overline{H}}^*)$$

$$= -j\omega\mu|\hat{\overline{H}}|^2 + j\omega\varepsilon(\omega)|\hat{\overline{E}}|^2$$

$$= -j\omega[\mu|\hat{\overline{H}}|^2 - \varepsilon(\omega)|\hat{\overline{E}}|^2]$$

$$\nabla\cdot[\tfrac{1}{2}\hat{\overline{E}} \times \hat{\overline{H}}^*] + 2j\omega[\tfrac{1}{4}\mu|\hat{\overline{H}}|^2 - \tfrac{1}{4}\varepsilon(\omega)|\hat{\overline{E}}|^2] = 0$$

16. a) and b) $m\frac{dv_x}{dt} = q(E_x + v_yB_z) \rightarrow j\omega\hat{v}_x = \frac{q}{m}\hat{E}_x + \omega_o\hat{v}_y;\ \omega_o = \frac{qB_z}{m}$

$$m\frac{dv_y}{dt} = q(E_y - v_xB_z) \rightarrow j\omega\hat{v}_y = \frac{q}{m}\hat{E}_y - \omega_o\hat{v}_x$$

$$m\frac{dv_z}{dt} = qE_z \rightarrow j\omega\hat{v}_z = \frac{q}{m}\hat{E}_z$$

$$\hat{v}_x = \frac{(\frac{q}{m})[j\omega\hat{E}_x + \omega_o\hat{E}_y]}{\omega_o^2 - \omega^2};\ \hat{v}_y = \frac{(\frac{q}{m})[j\omega\hat{E}_y - \omega_o\hat{E}_x]}{\omega_o^2 - \omega^2};\ \hat{v}_z = \frac{q}{j\omega m}\hat{E}_z$$

$$\hat{J}_x = qn\hat{v}_x = \frac{\varepsilon\ \omega_p^2}{(\omega_o^2 - \omega^2)}[j\omega\hat{E}_x + \omega_o\hat{E}_y];\ \omega_p^2 = \frac{q^2n}{m\varepsilon}$$

$$\hat{J}_y = qn\hat{v}_y = \frac{\varepsilon\omega_p^2}{(\omega_o^2 - \omega^2)}[j\omega\hat{E}_y - \omega_o\hat{E}_x]$$

$$\hat{J}_z = qn\hat{v}_z = \frac{\varepsilon\omega_p^2}{j\omega}\hat{E}_z$$

c) $\nabla \times \overline{E} = -\mu\frac{\partial\overline{H}}{\partial t} \rightarrow \begin{cases} \frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t} \rightarrow jk\hat{E}_x = j\omega\mu\hat{H}_y \\ \frac{\partial E_y}{\partial z} = \mu\frac{\partial H_x}{\partial t} \rightarrow -jk\hat{E}_y = j\omega\mu\hat{H}_x \end{cases}$

$$\nabla \times \overline{H} = \overline{J} + \varepsilon \frac{\partial \overline{E}}{\partial t} \rightarrow \begin{cases} \dfrac{\partial H_y}{\partial z} = -J_x - \varepsilon \dfrac{\partial E_x}{\partial t} \rightarrow jk\hat{H}_y = j\omega\varepsilon\hat{E}_x + \dfrac{\varepsilon\omega_p^2}{(\omega_o^2-\omega^2)} [j\omega\hat{E}_x + \omega_o\hat{E}_y] \\ \dfrac{\partial H_x}{\partial z} = J_y + \varepsilon \dfrac{\partial E_y}{\partial t} \rightarrow -jk\hat{H}_x = j\omega\varepsilon\hat{E}_y + \dfrac{\varepsilon\omega_p^2}{(\omega_o^2-\omega^2)} [j\omega\hat{E}_y - \omega_o\hat{E}_x] \end{cases}$$

$$jk\hat{H}_y = \frac{jk^2}{\omega\mu} \hat{E}_x = j\omega\varepsilon\hat{E}_x \left[1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}\right] + \frac{\varepsilon\omega_p^2\omega_o}{(\omega_o^2 - \omega^2)} \hat{E}_y$$

$$-jk\hat{H}_x = \frac{jk^2}{\omega\mu} \hat{E}_y = j\omega\varepsilon\hat{E}_y \left[1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2}\right] - \frac{\varepsilon\omega_p^2\omega_o}{(\omega_o^2 - \omega^2)} \hat{E}_x$$

$$\hat{E}_x \left[\frac{k^2c^2}{\omega^2} - (1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2})\right] + \frac{j\omega_p^2\omega_o}{(\omega_o^2 - \omega^2)\omega} \hat{E}_y = 0; \quad c^2 = \frac{1}{\varepsilon\mu}$$

$$-\hat{E}_x \frac{j\omega_p^2\omega_o}{\omega(\omega_o^2 - \omega^2)} + \hat{E}_y \left[\frac{k^2c^2}{\omega^2} - (1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2})\right] = 0$$

$$\left[\frac{k^2c^2}{\omega^2} - (1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2})\right]^2 - \left[\frac{\omega_p^2\omega_o}{\omega(\omega_o^2 - \omega^2)}\right]^2 = 0$$

$$k^2 = \frac{\omega^2}{c^2}\left\{1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2} \pm \frac{\omega_p^2\omega_o}{\omega(\omega_o^2 - \omega^2)}\right\}$$

$$= \frac{\omega^2}{c^2}\left\{1 + \frac{\omega_p^2}{(\omega_o^2 - \omega^2)} (1 \pm \frac{\omega_o}{\omega})\right\}$$

$$= \frac{\omega^2}{c^2}\left\{1 + \frac{\omega_p^2}{(\omega_o^2 - \omega^2)} (\frac{\omega \pm \omega_o}{\omega})\right\}$$

$$= \frac{\omega^2}{c^2}\left\{1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_o)}\right\}$$

d) $k = 0 \rightarrow \omega^2 \mp \omega\omega_o - \omega_p^2 = 0 \rightarrow \omega = \pm \frac{\omega_o}{2} \pm \sqrt{(\frac{\omega_o}{2})^2 + \omega_p^2}$

$k \to \infty \to \omega = \pm\, \omega_o$

Mode (1) $k^2 = \dfrac{\omega^2}{c^2}\left\{\dfrac{\omega^2 - \omega\omega_o - \omega_p^2}{\omega(\omega - \omega_o)}\right\}$

Mode (2) $k^2 = \dfrac{\omega^2}{c^2}\left\{\dfrac{\omega^2 + \omega\omega_o - \omega_p^2}{\omega(\omega + \omega_o)}\right\}$

Propagation $\to$ k real $\to$
- Mode (1) $\omega > \dfrac{\omega_o}{2} + \sqrt{(\dfrac{\omega_o}{2})^2 + \omega_p^2}, \qquad \omega < \omega_o$
- Mode (2) $\omega > -\dfrac{\omega_o}{2} + \sqrt{(\dfrac{\omega_o}{2})^2 + \omega_p^2}$

Evanescence $\to$ k imaginary $\to$
- Mode (1) $\omega_o < \omega < \dfrac{\omega_o}{2} + \sqrt{(\dfrac{\omega_o}{2})^2 + \omega_p^2}$
- Mode (2) $0 < \omega < -\dfrac{\omega_o}{2} + \sqrt{(\dfrac{\omega_o}{2})^2 + \omega_p^2}$

e) $\dfrac{E_y}{E_x} = \dfrac{j\omega_p^2\omega_o}{\omega(\omega_o^2 - \omega^2)\left[\dfrac{k^2c^2}{\omega^2} - (1 + \dfrac{\omega_p^2}{\omega_o^2 - \omega^2})\right]} = \pm\, j$ [Circular polarization]

f) $v_p = \dfrac{\omega}{k} = \dfrac{c}{[1 - \dfrac{\omega_p^2}{\omega(\omega \mp \omega_o)}]}$

$$E_o e^{j(\omega t - k_o z)}\bar{i}_x = \{\frac{E_o}{2}[\bar{i}_x + j\bar{i}_y] + \frac{E_o}{2}[\bar{i}_x - j\bar{i}_y]\}e^{j(\omega t - k_o z)}$$

g) $\bar{E}_T = \{E_1[\bar{i}_x + j\bar{i}_y]e^{j(\omega t - k_1 z)} + E_2[\bar{i}_x - j\bar{i}_y]e^{j(\omega t - k_2 z)}\}$

$$k_1 = \frac{\omega^2}{c^2}[1 - \frac{\omega_p^2}{\omega(\omega - \omega_o)}]$$

$$k_2 = \frac{\omega^2}{c^2}[1 - \frac{\omega_p^2}{\omega(\omega + \omega_o)}]$$

$$\hat{\overline{E}}_T(z=0) = E_t\overline{i}_x = E_1(\overline{i}_x + j\overline{i}_y) + E_2(\overline{i}_x - j\overline{i}_y) \rightarrow E_1 = E_2 = \frac{E_t}{2}$$

$$\overline{E}_T = \frac{E_t}{2}\{(\overline{i}_x + j\overline{i}_y)e^{-jk_1z} + (\overline{i}_x - j\overline{i}_y)e^{-jk_2z}\}e^{j\omega t}$$

$$\overline{H}_T = \frac{E_t}{2\omega\mu}\{k_1(\overline{i}_x - j\overline{i}_y)e^{-jk_1z} + k_2(\overline{i}_x + j\overline{i}_y)e^{-jk_2z}\}e^{j\omega t}$$

17. a) $\Delta\phi = \omega\ell\left(\frac{1}{c_{||}} - \frac{1}{c_{\perp}}\right) = \omega\ell\lambda BE_o^2 = 2\pi\ell cBE_o^2$

b) $\Delta\phi = \frac{\pi}{2} = 2\pi\ell cBE_o^2 \rightarrow E_o^2 = \frac{1}{4B\ell c}$

$\Delta\phi = \pi = 2\pi\ell cBE_o^2 \rightarrow E_o^2 = \frac{1}{2B\ell c}$

c) $\overline{E} = E_1 \text{Re}\, e^{j(\omega t - k_{||}\ell)}[\overline{i}_x + e^{j\phi}\overline{i}_y]; \ \phi = (k_{||} - k_{\perp})\ell$

Transmission axis of crossed polarizer in direction $\overline{i}_x - \overline{i}_y$

$$E_t = \overline{E}\cdot\frac{(\overline{i}_x - \overline{i}_y)}{\sqrt{2}} = \frac{E_1}{\sqrt{2}}\text{Re}\, e^{j(\omega t - k_{||}\ell)}[1 - e^{j\phi}]$$

$$S_t = \frac{1}{2}\frac{|E_t|^2}{\eta}$$

$$= \frac{1}{4}\frac{E_1^2}{\eta}[(1 - \cos\phi)^2 + \sin^2\phi]$$

$$= \frac{1}{2}\frac{E_1^2}{\eta}(1 - \cos\phi)$$

$$= \frac{E_1^2}{\eta}\sin^2\frac{\phi}{2}$$

Max light $\rightarrow \phi = (2n + 1)\pi \qquad n = 0, 1, 2, \ldots$

Min light $\rightarrow \phi = 2n\pi \qquad n = 1, 2, \ldots$

Section 7.5

18. a) $\overline{E}_i = \text{Re}\hat{E}_i e^{-jk_o z}\overline{i}_y$, $\overline{E}_r = \text{Re}\hat{E}_r e^{+jk_o z}\overline{i}_y$, $\overline{E}_t = \text{Re}\hat{E}_t e^{-jk_t z}\overline{i}_y$

$$\overline{H}_i = \text{Re}\,\frac{-\hat{E}_i}{\eta}\, e^{-jk_o z}\,\overline{i}_x,\quad \overline{H}_r = \text{Re}\,\frac{\hat{E}_r}{\eta}\, e^{+jk_o z}\,\overline{i}_x,\quad \overline{H}_t = \text{Re}\,\frac{-k_t}{\omega\mu}\,\hat{E}_t e^{-jk_t z}\,\overline{i}_x$$

$$k_o = \omega\sqrt{\varepsilon_o\mu_o},\quad k_t = \frac{\sqrt{\omega^2 - \omega_p^2}}{c}$$

b) $\hat{E}_i + \hat{E}_r = \hat{E}_t$

$$-\hat{E}_i + \hat{E}_r = \frac{-k_t\eta}{\omega\mu}\,\hat{E}_t \longrightarrow \hat{E}_r = \frac{\hat{E}_i[1 - \frac{k_t\eta}{\omega\mu}]}{1 + \frac{k_t\eta}{\omega\mu}},\quad \hat{E}_t = \frac{2\hat{E}_i}{1 + \frac{k_t\eta}{\omega\mu}}$$

c) $\langle S\rangle_z = -\frac{1}{2}\,\text{Re}E_y H_x^*$

$z > 0$ (Transmitted)

$$\langle S\rangle_z = \frac{1}{2}\,\text{Re}\left\{|\hat{E}_t|^2\,\frac{k_t^*}{\omega\mu}\,e^{-j(k_t - k_t^*)z}\right\}$$

$$= \begin{cases} \dfrac{1}{2}\,\dfrac{|\hat{E}_t|^2 k_t}{\omega\mu} = \dfrac{2|\hat{E}_i|^2 k_t}{\omega\mu[1 + \frac{k_t\eta}{\omega\mu}]^2} & \omega > \omega_p \ (k_t \text{ real}) \\ 0 & \omega < \omega_p \ (k_t \text{ imaginary}) \end{cases}$$

$z < 0$

$$\langle S\rangle_z = -\frac{1}{2\eta}\,\text{Re}[\hat{E}_i e^{-jk_o z} + \hat{E}_r e^{jk_o z}][-\hat{E}_i^* e^{jk_o z} + \hat{E}_r^* e^{-jk_o z}]$$

$$= -\frac{1}{2\eta}\,\text{Re}\left\{-|\hat{E}_i|^2 + |\hat{E}_r|^2 - \hat{E}_r\hat{E}_i^* e^{2jk_o z} + \hat{E}_i\hat{E}_r^* e^{-2jk_o z}\right\}$$

$$= \frac{1}{2\eta}\,[|\hat{E}_i|^2 - |\hat{E}_r|^2]$$

$$= \frac{|\hat{E}_i|^2}{2\eta}\left\{1 - \frac{|1 - \frac{k_t\eta}{\omega\mu}|^2}{|1 + \frac{k_t\eta}{\omega\mu}|^2}\right\}$$

$$k_t \text{ imaginary} \rightarrow |1 - \frac{k_t\eta}{\omega\mu}| = |1 + \frac{k_t\eta}{\omega\mu}| \rightarrow \langle S\rangle_z = 0$$

$$k_t \text{ real} \rightarrow \langle S\rangle_z = \frac{|\hat{E}_i|^2}{2\eta}\frac{4k_t\eta}{\omega\mu(1 + \frac{k_t\eta}{\omega\mu})^2} = \frac{2k_t|\hat{E}_i|^2}{\omega\mu[1 + \frac{k_t\eta}{\omega\mu}]^2}$$

19. a)

$$\bar{E} = E_x\cos(\omega t - kz)\bar{i}_x + E_y\sin(\omega t - kz)\bar{i}_y$$

$$\bar{H} = \frac{1}{\eta}\left[E_x\cos(\omega t - kz)\bar{i}_y - E_y\sin(\omega t - kz)\bar{i}_x\right]$$

$$\bar{S}_i = \bar{E}\times\bar{H} = \frac{1}{\eta}\left[E_x^2\cos^2(\omega t - kz) + E_y^2\sin^2(\omega t - kz)\right]\bar{i}_z$$

b) $\bar{E}_t = E_x\cos(\omega t - kz)\bar{i}_x$

$$\bar{H}_t = \frac{E_x}{\eta}\cos(\omega t - kz)\bar{i}_y$$

$$\bar{S}_t = \frac{E_x^2}{\eta}\cos^2(\omega t - kz)\bar{i}_z$$

c) $|\bar{E}_{t2}| = |\bar{E}_t\cos\phi|$

$|\bar{H}_{t2}| = |\frac{\bar{E}_t\cos\phi}{\eta}|$ $\longrightarrow S_t = \frac{|\bar{E}_t|^2}{\eta}\cos^2\phi$ [Law of Malus obeyed]

20. a) $\bar{E}_i = \text{Re}E_o e^{j\omega(t - \frac{z}{c})}\bar{i}_y;\quad \bar{E}_r = \text{Re}\hat{E}_r e^{j\omega'(t + \frac{z}{c})}$

$$\bar{H}_i = \text{Re} - \frac{E_o}{\eta}e^{j\omega(t - \frac{z}{c})}\bar{i}_x;\quad \bar{H}_r = \text{Re}\frac{\hat{E}_r}{\eta}e^{j\omega'(t + \frac{z}{c})}$$

b) $E_T(z=vt) = 0 \rightarrow E_o e^{j\omega t(1 - \frac{v}{c})} + \hat{E}_r e^{j\omega' t(1 + \frac{v}{c})} = 0$

$$\omega(1-\frac{v}{c}) = \omega'(1+\frac{v}{c}) \rightarrow \omega' = \frac{\omega(1-\frac{v}{c})}{1+\frac{v}{c}} \approx \omega(1-\frac{v}{c})^2 \approx \omega(1-\frac{2v}{c})$$

$$\hat{E}_r = -E_o$$

$$\overline{E}_T = \overline{E}_i+\overline{E}_r = \text{Re}E_o[e^{j\omega(t-\frac{z}{c})} - e^{j\omega'(t+\frac{z}{c})}]\overline{i}_y = E_o[\cos\omega(t-\frac{z}{c}) - \cos\omega'(t+\frac{z}{c})]\overline{i}_y$$

$$\overline{H}_T = \overline{H}_i+\overline{H}_r = \text{Re}\,\frac{-E_o}{\eta}[e^{j\omega(t-\frac{z}{c})} + e^{j\omega'(t+\frac{z}{c})}]\overline{i}_x = \frac{-E_o}{\eta}[\cos\omega(t-\frac{z}{c}) + \cos\omega'(t+\frac{z}{c})]\overline{i}_x$$

c) $\overline{S} = \overline{E}_T \times \overline{H}_T = \frac{E_o^2}{\eta}[\cos^2\omega(t-\frac{z}{c}) - \cos^2\omega'(t+\frac{z}{c})]\overline{i}_z$

We cannot use the complex Poynting vector because the incident and reflected waves are at different frequencies.

Section 7.6

21. a) $\overline{E}_i = \text{Re}\hat{E}_i e^{j(\omega t-k_1 z)}\overline{i}_x;\ k_1 = \omega\sqrt{\varepsilon_1\mu_1}$

$$\overline{H}_i = \text{Re}\,\frac{\hat{E}_i}{\eta_1} e^{j(\omega t-k_1 z)}\overline{i}_y;\ \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\overline{E}_r = \text{Re}\hat{E}_r e^{j(\omega t+k_1 z)}\overline{i}_x$$

$$\overline{H}_r = \text{Re} - \frac{\hat{E}_r}{\eta} e^{j(\omega t+k_1 z)}\overline{i}_y$$

$$\overline{E}_t = \text{Re}\hat{E}_t \sin k_2(z-d)e^{j\omega t}\overline{i}_x;\quad k_2 = \omega\sqrt{\varepsilon_2\mu_2},\ \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

$$\overline{H}_t = \text{Re}[-\frac{1}{j\omega\mu}\frac{\partial E_t}{\partial z}]\overline{i}_y = \text{Re}\,\frac{-k_2}{j\omega\mu}\hat{E}_t\cos k_2(z-d)e^{j\omega t}\overline{i}_y = \text{Re}\,\frac{j\hat{E}_t}{\eta_2}\cos k_2(z-d)e^{j\omega t}\overline{i}_y$$

b) Boundary Conditions:

$$\overline{E}_i(z=0) + \overline{E}_r(z=0) = \overline{E}_t(z=0) \rightarrow \hat{E}_i + \hat{E}_r = -\hat{E}_t\sin k_2 d$$

$$\overline{H}_i(z=0) + \overline{H}_r(z=0) = \overline{H}_t(z=0) \rightarrow \hat{E}_i - \hat{E}_r = \frac{j\eta_1}{\eta_2}\hat{E}_t \cos k_2 d$$

$$\hat{E}_r = \frac{-\hat{E}_i[j\eta_1 \cos k_2 d + \eta_2 \sin k_2 d]}{[j\eta_1 \cos k_2 d - \eta_2 \sin k_2 d]}; \quad \hat{E}_t = \frac{2\eta_2 \hat{E}_i}{j\eta_1 \cos k_2 d - \eta_2 \sin k_2 d}$$

c) $<\overline{S}> = \frac{1}{2}\,\mathrm{Re}\,\hat{\overline{E}} \times \hat{\overline{H}}^*$

$z < 0$

$$<S>_z = \frac{1}{2\eta_1}\,\mathrm{Re}[\hat{E}_i e^{-jk_1 z} + \hat{E}_r e^{jk_1 z}][\hat{E}_i^* e^{jk_1 z} - \hat{E}_r^* e^{-jk_1 z}]$$

$$= \frac{1}{2\eta_1}[|\hat{E}_i|^2 - |\hat{E}_r|^2]$$

$$= 0$$

$z > 0$

$$<S>_z = \frac{1}{2\eta_2}\,\mathrm{Re}[\hat{E}_t \sin k_2(z-d)(-j)\hat{E}_t^* \cos k_2(z-d)]$$

$$= 0$$

d) $\hat{\overline{K}}(z=d) = \hat{H}_y(z=d)\overline{i}_x = \frac{j\hat{E}_t}{\eta_2}\,\overline{i}_x$

$$F_z = \frac{1}{4}\,\mathrm{Re}\,\mu_2 \hat{K}_x \hat{H}_y^*$$

$$= \frac{1}{4}\mu_2 |\hat{K}_x|^2$$

$$= \frac{1}{4}\mu_2 \frac{|\hat{E}_t|^2}{\eta_2^2} = \frac{1}{4}\varepsilon_2 |\hat{E}_t|^2 = \frac{\varepsilon_2 \eta_2^2 |\hat{E}_i|^2}{[\eta_1^2 \cos^2 k_2 d + \eta_2^2 \sin^2 k_2 d]}$$

Section 7.7

22. $\overline{E} = \text{Re}\ \hat{\overline{E}} e^{j\omega t} e^{-\overline{\gamma}\cdot\overline{r}}$ $\longrightarrow$ $-\overline{\gamma} \times \hat{\overline{E}} = -j\omega\mu\hat{\overline{H}}$

$\overline{H} = \text{Re}\ \hat{\overline{H}} e^{j\omega t}\ e^{-\overline{\gamma}\cdot\overline{r}}$ $\qquad$ $-\overline{\gamma} \times \hat{\overline{H}} = j\omega\varepsilon(1 + \frac{\sigma}{j\omega\varepsilon})\hat{\overline{E}}$

$$-\overline{\gamma}\cdot\varepsilon\hat{\overline{E}} = 0, \quad -\overline{\gamma}\cdot\mu\hat{\overline{H}} = 0$$

$$\overline{\gamma}\cdot\overline{\gamma} = (\alpha^2 - k^2 + 2j\overline{\alpha}\cdot\overline{k}) = -\omega^2\mu\varepsilon(1 + \frac{\sigma}{j\omega\varepsilon})$$

$$\alpha^2 - k^2 = -\omega^2\mu\varepsilon$$

$$\overline{\alpha}\cdot\overline{k} = \frac{\sigma\omega\mu}{2} = \frac{1}{\delta^2}$$

23. a) $$\overline{H} = \begin{cases} \text{Re}\ \hat{H}_1 e^{j\omega t} e^{-\overline{\gamma}_1\cdot\overline{r}}\ \overline{i}_y \\ \text{Re}\ \hat{H}_2 e^{j\omega t} e^{-\overline{\gamma}_2\cdot\overline{r}}\ \overline{i}_y \end{cases} ; \qquad \overline{E} = \begin{cases} \text{Re} - \frac{\overline{\gamma}_1 \times \hat{H}_1 \overline{i}_y}{j\omega\varepsilon}\ e^{j\omega t} e^{-\overline{\gamma}_1\cdot\overline{r}} & z > 0 \\ \text{Re} - \frac{\overline{\gamma}_2 \times \hat{H}_2 \overline{i}_y}{j\omega\varepsilon}\ e^{j\omega t} e^{-\overline{\gamma}_2\cdot\overline{r}} & z < 0 \end{cases}$$

$$\overline{\gamma}_1 = jk_x\overline{i}_x + \gamma_z\overline{i}_z$$

$$\longrightarrow \quad \gamma_z = [k_x^2 - \frac{\omega^2}{c^2}]^{1/2}$$

$$\overline{\gamma}_2 = jk_x\overline{i}_x - \gamma_z\overline{i}_z$$

$$\overline{\gamma}_1 \times \overline{H}_1 = [jk_x\overline{i}_x + \gamma_z\overline{i}_z] \times H_1\overline{i}_y = H_1[jk_x\overline{i}_z - \gamma_z\overline{i}_x]$$

$$\overline{\gamma}_2 \times \overline{H}_2 = H_2[jk_x\overline{i}_z + \gamma_z\overline{i}_x]$$

$$\hat{\overline{E}} = \begin{cases} -\frac{\hat{H}_1}{j\omega\varepsilon}[jk_x\overline{i}_z - \gamma_z\overline{i}_x]e^{j\omega t} e^{-\overline{\gamma}_1\cdot\overline{r}} & z > 0 \\ -\frac{\hat{H}_2}{j\omega\varepsilon}[jk_x\overline{i}_z + \gamma_z\overline{i}_x]e^{j\omega t} e^{-\overline{\gamma}_2\cdot\overline{r}} & z < 0 \end{cases}$$

$$E_x(z=0_+) = E_x(z=0_-) \rightarrow \hat{H}_1 = -\hat{H}_2$$

$$\varepsilon(E_z(z=0_+) - E_z(z=0_-)) = \sigma_o \sin(\omega t - k_x x) \rightarrow \frac{k_x}{\omega}[-\hat{H}_1 + \hat{H}_2] = -j\sigma_o$$

$$\hat{H}_1 = -\hat{H}_2 = \frac{j\sigma_o \omega}{2k_x}$$

b) $\hat{K}_x(z=0) = -\hat{H}_y(z=0_+) + \hat{H}_y(z=0_-) = -\hat{H}_1 + \hat{H}_2 = -\dfrac{j\sigma_o\omega}{k_x}$

Check: $\nabla_\Sigma \cdot \overline{K} + \dfrac{\partial \sigma_f}{\partial t} = 0 \rightarrow \dfrac{d\hat{K}_x}{dx} + j\omega\hat{\sigma}_o = 0 \rightarrow \hat{K}_x = \dfrac{-j\omega\hat{\sigma}_o}{-jk_x} = \dfrac{\omega\hat{\sigma}_o}{k_x}$

$$\hat{\sigma}_o = -j\sigma_o \rightarrow \hat{K}_x = \frac{-j\omega\sigma_o}{k_x}$$

24. a) $\gamma_{z1} = [k_x^2 - \dfrac{\omega^2}{c_o^2}]^{1/2}$; $c_o = \dfrac{1}{\sqrt{\varepsilon_o \mu_o}}$ Non-uniform $\rightarrow k_x > \dfrac{\omega}{c_o}$

$\rightarrow c < \dfrac{\omega}{k_x} < c_o$

$\gamma_{z2} = [k_x^2 - \dfrac{\omega^2}{c^2}]^{1/2}$; $c = \dfrac{1}{\sqrt{\varepsilon\mu}}$ Uniform $\rightarrow k_x < \dfrac{\omega}{c}$

Non-uniform in each region $\rightarrow k_x > \dfrac{\omega}{c} \rightarrow \omega < k_x c$

Uniform in each region $\rightarrow \omega > k_x c_o$

b)
$$\hat{H}_y = \begin{cases} \hat{H}_1 e^{-jk_x x} e^{\gamma_{z1} z} & z < 0 \\ [\hat{H}_2 e^{-\gamma_{z1} z} + \hat{H}_3 e^{\gamma_{z1} z}] e^{-jk_x x} & 0 < z < d \\ \hat{H}_4 e^{-\gamma_{z2}(z-d)} e^{-jk_x x} & z > d \end{cases}$$

$$\hat{\overline{E}} = \frac{\nabla \times \hat{\overline{H}}}{j\omega\varepsilon} = \begin{cases} -\dfrac{\hat{H}_1}{j\omega\varepsilon_o}[jk_x \overline{i}_z + \gamma_{z1}\overline{i}_x] e^{-jk_x x} e^{\gamma_{z1} z} & z < 0 \\ -\dfrac{1}{j\omega\varepsilon_o}\{[jk_x \overline{i}_z - \gamma_{z1}\overline{i}_x]\hat{H}_2 e^{-\gamma_{z1} z} + [jk_x \overline{i}_z + \gamma_{z1}\overline{i}_x]\hat{H}_3 e^{\gamma_{z1} z}\} e^{-jk_x x} & 0 < z < d \\ -\dfrac{\hat{H}_4}{j\omega\varepsilon}[jk_x \overline{i}_z - \gamma_{z2}\overline{i}_x] e^{-\gamma_{z2}(z-d)} e^{-jk_x x} & z > d \end{cases}$$

Boundary Conditions:

$$H_y(z=0_-) - H_y(z=0_+) = K_o \rightarrow \hat{H}_1 - \hat{H}_2 - \hat{H}_3 = K_o$$

$$E_x(z=0_-) = E_x(z=0_+) \rightarrow -\hat{H}_1 = \hat{H}_2 - \hat{H}_3$$

$$H_y(z=d_-) = H_y(z=d_+) \rightarrow \hat{H}_2 e^{-\gamma_{z1}d} + \hat{H}_3 e^{\gamma_{z1}d} = \hat{H}_4$$

$$E_x(z=d_-) = E_x(z=d_+) \rightarrow -\hat{H}_2 e^{-\gamma_{z1}d} + \hat{H}_3 e^{\gamma_{z1}d} = -\frac{\gamma_{z2}}{\gamma_{z1}}\frac{\varepsilon_o}{\varepsilon}\hat{H}_4$$

$$\hat{H}_1 = \frac{K_o}{2}\left[\frac{1 + \frac{\varepsilon_o}{\varepsilon}\frac{\gamma_{z2}}{\gamma_{z1}} - (1 - \frac{\varepsilon_o}{\varepsilon}\frac{\gamma_{z2}}{\gamma_{z1}})e^{-2\gamma_{z1}d}}{1 + \frac{\varepsilon_o}{\varepsilon}\frac{\gamma_{z2}}{\gamma_{z1}}}\right]; \quad \hat{H}_2 = -\frac{K_o}{2}$$

$$\hat{H}_3 = \frac{-\frac{K_o}{2} e^{-2\gamma_{z1}d}(1 - \frac{\varepsilon_o}{\varepsilon}\frac{\gamma_{z2}}{\gamma_{z1}})}{1 + \frac{\varepsilon_o}{\varepsilon}\frac{\gamma_{z2}}{\gamma_{z1}}}; \quad \hat{H}_4 = \frac{-K_o e^{-\gamma_{z1}d}}{1 + \frac{\varepsilon_o}{\varepsilon}\frac{\gamma_{z2}}{\gamma_{z1}}}$$

c) γ_{z1} real (non-uniform), γ_{z2} imaginary (uniform)

$z < 0$

$$<\overline{S}> = \frac{1}{2}\,\mathrm{Re}\,\overline{E} \times \overline{H}^* = \frac{1}{2}\,\mathrm{Re}\left[\frac{-\hat{H}_1}{j\omega\varepsilon_o}[jk_x\overline{i}_z + \gamma_{z1}\overline{i}_x]e^{\gamma_{z1}z} \times \hat{H}_1^* e^{\gamma_{z1}^* z}\overline{i}_y\right]$$

$$= -\frac{|\hat{H}_1|^2}{2}\,\mathrm{Re}\left\{\frac{-jk_x\overline{i}_x + \gamma_{z1}\overline{i}_z}{j\omega\varepsilon_o} e^{(\gamma_{z1}+\gamma_{z1}^*)z}\right\}$$

$$= \frac{|\hat{H}_1|^2 k_x}{2\omega\varepsilon_o} e^{2\gamma_{z1}z}\,\overline{i}_x$$

$$<S_z> = 0$$

$z > d$

$$<\overline{S}> = \frac{1}{2}\,\mathrm{Re}\left[\frac{-\hat{H}_4}{j\omega\varepsilon}[jk_x\overline{i}_z - \gamma_{z2}\overline{i}_x]e^{-\gamma_{z2}(z-d)} \times \hat{H}_4^* e^{-\gamma_{z2}^*(z-d)}\overline{i}_y\right]$$

$$= -\frac{1}{2}|\hat{H}_4|^2\,\mathrm{Re}\left[\frac{-jk_x\overline{i}_x - \gamma_{z2}\overline{i}_z}{j\omega\varepsilon} e^{-(z-d)(\gamma_{z2}+\gamma_{z2}^*)}\right]$$

$$= \frac{1}{2}|\hat{H}_4|^2 \frac{k_x\overline{i}_x + |\gamma_{z2}|\overline{i}_z}{\omega\varepsilon}$$

$$= \frac{K_o^2 e^{-2\gamma_{z1}d}\left[k_x\overline{i}_x + |\gamma_{z2}|\overline{i}_z\right]}{2[1 + (\frac{\varepsilon_o}{\varepsilon}\frac{|\gamma_{z2}|}{\gamma_{z1}})^2]\omega\varepsilon}$$

$$\langle S_z\rangle = \frac{K_o^2 e^{-2\gamma_{z1}d}|\gamma_{z2}|}{2\omega\varepsilon[1 + (\frac{\varepsilon_o}{\varepsilon}\frac{|\gamma_{z2}|}{\gamma_{z1}})^2]}$$

$0 < z < d$

$$\langle\overline{S}\rangle = \frac{1}{2}\,\mathrm{Re}\left\{\frac{\hat{H}_2 e^{-\gamma_{z1}z}(jk_x\overline{i}_z - \gamma_{z1}\overline{i}_x) + \hat{H}_3 e^{\gamma_{z1}z}(jk_x\overline{i}_z + \gamma_{z1}\overline{i}_x)}{j\omega\varepsilon_o} \times[\hat{H}_2^* e^{-\gamma_{z1}z} + \hat{H}_3^* e^{\gamma_{z1}z}]\overline{i}_y\right\}$$

$$= \frac{1}{2\omega\varepsilon_o}\,\mathrm{Re}\;-j\{|\hat{H}_2|^2 e^{-2\gamma_{z1}z}(-jk_x\overline{i}_x - \gamma_{z1}\overline{i}_z) + |\hat{H}_3|^2 e^{2\gamma_{z1}z}(-jk_x\overline{i}_x + \gamma_{z1}\overline{i}_z)$$

$$+ \hat{H}_2\hat{H}_3^*(-jk_x\overline{i}_x - \gamma_{z1}\overline{i}_z) + \hat{H}_3\hat{H}_2^*(-jk_x\overline{i}_x + \gamma_{z1}\overline{i}_z)\}$$

$$= \frac{1}{2\omega\varepsilon_o}\{-k_x[+|\hat{H}_2|^2 e^{-2\gamma_{z1}z} + |\hat{H}_3|^2 e^{2\gamma_{z1}z} + 2\mathrm{Re}(\hat{H}_2\hat{H}_3^*)]\overline{i}_x + 2\gamma_{z1}\mathrm{Re}\,j\hat{H}_2\hat{H}_3^*\overline{i}_z\}$$

$$\langle S_z\rangle = \frac{\gamma_{z1}}{\omega\varepsilon_o}\mathrm{Re}\,j\hat{H}_2\hat{H}_3^*$$

$$= \frac{\frac{\gamma_{z1}}{\omega\varepsilon_o}(\frac{K_o}{2})^2 e^{-2\gamma_{z1}d}\frac{2\varepsilon_o}{\varepsilon}\frac{|\gamma_{z2}|}{\gamma_{z1}}}{[1 + (\frac{\varepsilon_o}{\varepsilon}\frac{|\gamma_{z2}|}{\gamma_{z1}})^2]} = \frac{K_o^2 e^{-2\gamma_{z1}d}|\gamma_{z2}|}{2\omega\varepsilon[1 + (\frac{\varepsilon_o}{\varepsilon}\frac{|\gamma_{z2}|}{\gamma_{z1}})^2]}$$

$$\langle S_z\rangle\Big|_{0<z<d} = \langle S_z\rangle\Big|_{z>d}$$

Section 7.8

25. a) $\overline{E}_i = \mathrm{Re}\hat{E}_i e^{j(\omega t - k_{xi}x - k_{zi}z)}\overline{i}_y;\quad k_{xi} = k_o\sin\theta$

$$\overline{H}_i = \mathrm{Re}\,\frac{\hat{E}_i}{\eta}[-\cos\theta\overline{i}_x + \sin\theta\overline{i}_z]e^{j(\omega t - k_{xi}x - k_{zi}z)};\quad k_{zi} = k_o\cos\theta$$

$$\overline{E}_r = \mathrm{Re}\hat{E}_r e^{j(\omega t + k_{xi}x + k_{zi}z)}\overline{i}_y \; ; \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\overline{H}_r = \mathrm{Re}\,\frac{\hat{E}_r}{\eta}\,[\cos\theta\overline{i}_x - \sin\theta\overline{i}_z]e^{j(\omega t + k_{xi}x + k_{zi}z)}$$

$$\overline{E}_1 = \mathrm{Re}\hat{E}_1 e^{j(\omega t - k_{xi}x + k_{zi}z)}\overline{i}_y$$

$$\overline{H}_1 = \mathrm{Re}\,\frac{\hat{E}_1}{\eta}\,[\cos\theta\overline{i}_x + \sin\theta\overline{i}_z]e^{j(\omega t - k_{xi}x + k_{zi}z)}$$

$$\overline{E}_2 = \mathrm{Re}\hat{E}_2 e^{j(\omega t + k_{xi}x - k_{zi}z)}\overline{i}_y$$

$$\overline{H}_2 = \mathrm{Re}\,\frac{\hat{E}_2}{\eta}\,[-\cos\theta\overline{i}_x - \sin\theta\overline{i}_z]e^{j(\omega t + k_{xi}x - k_{zi}z)}$$

b) $$\overline{E}_T(z=0) = 0 \rightarrow (\hat{E}_i + \hat{E}_1)e^{-jk_{xi}x} + (\hat{E}_r + \hat{E}_2)e^{jk_{xi}x} = 0$$

$$\hat{E}_i + \hat{E}_1 = 0,\ \hat{E}_r + \hat{E}_2 = 0$$

$$\overline{E}_T(x=0) = 0 \rightarrow (\hat{E}_i + \hat{E}_2)e^{-jk_{zi}z} + (\hat{E}_r + \hat{E}_1)e^{jk_{zi}z} = 0$$

$$\hat{E}_i + \hat{E}_2 = 0,\ \hat{E}_r + \hat{E}_1 = 0 \rightarrow \hat{E}_1 = -\hat{E}_i,\ \hat{E}_2 = -\hat{E}_i,\ \hat{E}_r = \hat{E}_i$$

$$\hat{\overline{E}}_T(x,z) = \hat{E}_i\{e^{-j(k_{xi}x+k_{zi}z)} + e^{j(k_{xi}x+k_{zi}z)} - e^{-j(k_{xi}x - k_{zi}z)} - e^{j(k_{xi}x-k_{zi}z)}\}\overline{i}_y$$

$$= 2\hat{E}_i\{\cos(k_{xi}x + k_{zi}z) - \cos(k_{xi}x - k_{zi}z)\}\overline{i}_y$$

$$= -4\hat{E}_i \sin k_{xi}x \sin k_{zi}z\ \overline{i}_y$$

$$\overline{E}_T(x,z,t) = \mathrm{Re}\hat{\overline{E}}_T(x,z)e^{j\omega t} = -4E_o \sin k_{xi}x \sin k_{zi}z \cos\omega t\ \overline{i}_y$$

$$\hat{\overline{H}}_T(x,z) = \frac{\hat{E}_i}{\eta}\{-\cos\theta\overline{i}_x[e^{-j(k_{xi}x+k_{zi}z)} - e^{j(k_{xi}x+k_{zi}z)} + e^{-j(k_{xi}x-k_{zi}z)} - e^{j(k_{xi}x-k_{zi}z)}]$$

$$+\sin\theta\overline{i}_z[e^{-j(k_{xi}x+k_{zi}z)} - e^{j(k_{xi}x+k_{zi}z)} - e^{-j(k_{xi}x-k_{zi}z)} + e^{j(k_{xi}x-k_{zi}z)}]\}$$

$$= 2j\frac{\hat{E}_i}{\eta}\{\cos\theta\overline{i}_x[\sin(k_{xi}x+k_{zi}z)+\sin(k_{xi}x-k_{zi}z)]$$

$$-\sin\theta\overline{i}_z[\sin(k_{xi}x+k_{zi}z)-\sin(k_{xi}x-k_{zi}z)]\}$$

$$= 4j\frac{\hat{E}_i}{\eta}\{\cos\theta\sin k_{xi}x\cos k_{zi}z\overline{i}_x - \sin\theta\cos k_{xi}x\sin k_{zi}z\overline{i}_z\}$$

$$\overline{H}_T(x,z,t) = \mathrm{Re}\hat{\overline{H}}_T(x,z)e^{j\omega t} = \frac{-4E_o}{\eta}\{\cos\theta\sin k_{xi}x\cos k_{zi}z\overline{i}_x - \sin\theta\cos k_{xi}x\sin k_{zi}z\overline{i}_z\}\sin\omega t$$

c) $\hat{\overline{K}}(z=0) = -\hat{H}_x(z=0)\overline{i}_y = \frac{-4j\hat{E}_i}{\eta}\cos\theta\sin k_{xi}x$

$\hat{\overline{K}}(x=0) = \hat{H}_z(x=0)\overline{i}_y = \frac{-4j\hat{E}_i}{\eta}\sin\theta\sin k_{zi}z$

$\hat{\sigma}_f(x=0) = \hat{\sigma}_f(z=0) = 0$

d) $F_z(z=0) = \frac{1}{2}\mu_o|\hat{\overline{K}}(z=0)|^2 = \frac{8\mu_o}{\eta^2}|\hat{E}_i|^2\cos^2\theta\sin^2 k_{xi}x = 8\varepsilon_o|\hat{E}_i|^2\cos^2\theta\sin^2 k_{xi}x$

$F_x(z=0) = \frac{1}{2}\mu_o|\hat{\overline{K}}(x=0)|^2 = 8\varepsilon_o|\hat{E}_i|^2\sin^2\theta\sin^2 k_{zi}z$

e) $\overline{S} = \overline{E}\times\overline{H}$

$$= \frac{-16E_o^2}{\eta}\sin k_{xi}x\sin k_{zi}z\sin\omega t\cos\omega t\{\cos\theta\sin k_{xi}x\cos k_{zi}z\overline{i}_z + \sin\theta\cos k_{xi}x\sin k_{zi}z\overline{i}_x\}$$

<u>Section 7.9</u>

26. a) $\cos\theta_i = \frac{h_1}{s_i} \rightarrow s_i = \frac{h_1}{\cos\theta_i}$

$\cos\theta_r = \frac{h_2}{s_r} \rightarrow s_r = \frac{h_2}{\cos\theta_r}$

$\longrightarrow t = \frac{s_i+s_r}{c_1} = \frac{1}{c_1}\left[\frac{h_1}{\cos\theta_i}+\frac{h_2}{\cos\theta_r}\right]$

$\sin\theta_i = \frac{L_1}{s_i},\ \sin\theta_r = \frac{L_2}{s_r} \rightarrow L_1 + L_2 = L_{AB} = s_i\sin\theta_i + s_r\sin\theta_r$

$$= h_1\tan\theta_i + h_2\tan\theta_r$$

b) $0 = \frac{h_1}{\cos^2\theta_i}\, d\theta_i + \frac{h_2}{\cos^2\theta_r}\, d\theta_r \rightarrow \frac{d\theta_r}{d\theta_i} = -\frac{h_1}{h_2}\frac{\cos^2\theta_r}{\cos^2\theta_i}$

$$dt = 0 = \frac{1}{c_1}\left[\frac{h_1\sin\theta_i d\theta_i}{\cos^2\theta_i} + \frac{h_2\sin\theta_r d\theta_r}{\cos^2\theta_r}\right] \rightarrow \frac{d\theta_r}{d\theta_i} = \frac{-h_1}{h_2}\frac{\sin\theta_i}{\sin\theta_r}\frac{\cos^2\theta_r}{\cos^2\theta_i}$$

$$\rightarrow \sin\theta_i = \sin\theta_r$$

c) $t = \frac{1}{c_1}\frac{h_1}{\cos\theta_i} + \frac{1}{c_2}\frac{h_2}{\cos\theta_t}$

$$L_{AC} = h_1\tan\theta_i + h_2\tan\theta_t$$

d) $0 = \frac{h_1}{\cos^2\theta_i}\, d\theta_i + \frac{h_2}{\cos^2\theta_t}\, d\theta_t \rightarrow \frac{d\theta_t}{d\theta_i} = \frac{-h_1}{h_2}\frac{\cos^2\theta_t}{\cos^2\theta_i}$

$$dt = 0 = \frac{h_1}{c_1\cos^2\theta_i}\sin\theta_i d\theta_i + \frac{h_2}{c_2\cos^2\theta_t}\sin\theta_t d\theta_t \rightarrow \frac{d\theta_t}{d\theta_i} = \frac{-h_1 c_2}{h_2 c_1}\frac{\sin\theta_i\cos^2\theta_t}{\sin\theta_t\cos^2\theta_i}$$

$$\rightarrow c_2\sin\theta_i = c_1\sin\theta_t \qquad \text{[Snell's law]}$$

27. $\mu = \mu_o,\ \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_o}{\varepsilon_1}};\quad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_o}{\varepsilon_2}}$

<u>$\overline{E}$ ∥ interface</u> $\qquad \left(\sin\theta_t = \frac{c_2}{c_1}\sin\theta_i = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i\right)$

$$R = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \left(\frac{\cos\theta_i}{\sqrt{\varepsilon_2}} - \frac{\cos\theta_t}{\sqrt{\varepsilon_1}}\right)\Big/\left(\frac{\cos\theta_i}{\sqrt{\varepsilon_2}} + \frac{\cos\theta_t}{\sqrt{\varepsilon_1}}\right)$$

$$= \frac{\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i\cos\theta_i - \sin\theta_i\cos\theta_t}{\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i\cos\theta_i + \sin\theta_i\cos\theta_t}$$

$$= \frac{\sin\theta_t \cos\theta_i - \sin\theta_i \cos\theta_t}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t}$$

$$= \frac{-\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$T = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{\frac{2\cos\theta_i}{\sqrt{\varepsilon_2}}}{\frac{\cos\theta_i}{\sqrt{\varepsilon_2}} + \frac{\cos\theta_t}{\sqrt{\varepsilon_1}}} = \frac{2\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\, \sin\theta_i \cos\theta_i}{\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\, \sin\theta_i \cos\theta_i + \sin\theta_i \cos\theta_t}$$

$$= \frac{2\sin\theta_t \cos\theta_i}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t}$$

$$= \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)}$$

<u>$\overline{H}$ $\parallel$ interface</u>

$$R = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} = \left(\frac{\cos\theta_i}{\sqrt{\varepsilon_1}} - \frac{\cos\theta_t}{\sqrt{\varepsilon_2}}\right) \Big/ \left(\frac{\cos\theta_t}{\sqrt{\varepsilon_2}} + \frac{\cos\theta_i}{\sqrt{\varepsilon_1}}\right)$$

$$= \frac{\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\, \sin\theta_t \cos\theta_i - \sin\theta_t \cos\theta_t}{\sin\theta_t \cos\theta_t + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\, \sin\theta_t \cos\theta_i}$$

$$= \frac{\sin\theta_i \cos\theta_i - \sin\theta_t \cos\theta_t}{\sin\theta_t \cos\theta_t + \sin\theta_i \cos\theta_i}$$

$$= \frac{\sin(\theta_i - \theta_t)\cos(\theta_i + \theta_t)}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}$$

$$= \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$T = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\dfrac{2\cos\theta_i}{\sqrt{\varepsilon_2}}}{\dfrac{\cos\theta_t}{\sqrt{\varepsilon_2}} + \dfrac{\cos\theta_i}{\sqrt{\varepsilon_1}}}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{\sin\theta_t\cos\theta_t + \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}\cos\theta_i\ \sin\theta_t}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{\sin(\theta_t + \theta_i)\cos(\theta_i - \theta_t)}$$

28. $n = A + \dfrac{B}{\lambda^2}$; $A = 1.5$, $B = 5 \times 10^{-15}$

Color	λ (meters)	n	θ_t $[\sin\theta_t = \dfrac{\sin\theta_i}{n}]$
violet	4×10^{-7}	1.5313	19.06°
blue	4.5×10^{-7}	1.5247	19.14°
green	5.5×10^{-7}	1.5165	19.25°
yellow	6×10^{-7}	1.5139	19.29°
orange	6.5×10^{-7}	1.5118	19.31°
red	7×10^{-7}	1.5102	19.33°

29. a) $\bar{E}_i = \mathrm{Re}\hat{E}_i[\cos\theta_i\bar{i}_x - \sin\theta_i\bar{i}_z]e^{j(\omega t - k_{xi}x - k_{zi}z)}$; $k_{xi} = k_1\sin\theta_i$; $k_1 = \omega\sqrt{\varepsilon_1\mu_1}$

$$\bar{H}_i = \mathrm{Re}\,\frac{\hat{E}_i}{\eta_1}\,e^{j(\omega t - k_{xi}x - k_{zi}z)}\bar{i}_y;\quad k_{zi} = k_1\cos\theta_i,\ \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\bar{E}_r = \mathrm{Re}\hat{E}_r[-\cos\theta_i\bar{i}_x - \sin\theta_i\bar{i}_z]e^{j(\omega t - k_{xi}x + k_{zi}z)}$$

$$\bar{H}_r = \mathrm{Re}\,\frac{\hat{E}_r}{\eta_1}\,e^{j(\omega t - k_{xi}x + k_{zi}z)}\bar{i}_y$$

$$\overline{E}_1 = \mathrm{Re}\,\hat{E}_1[\cos\theta_t \overline{i}_x - \sin\theta_t \overline{i}_z]e^{j(\omega t - k_{xt}x - k_{zt}z)}\ ; \quad k_{xt} = k_2\sin\theta_t;\ k_2 = \omega\sqrt{\varepsilon_2\mu_2}$$

$$\overline{H}_1 = \mathrm{Re}\,\frac{\hat{E}_1}{\eta_2}e^{j(\omega t - k_{xt}x - k_{zt}z)}\ ; \quad k_{zt} = k_2\cos\theta_t;\ \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

$$\overline{E}_2 = \mathrm{Re}\,\hat{E}_2[-\cos\theta_t \overline{i}_x - \sin\theta_t \overline{i}_z]e^{j(\omega t - k_{xt}x + k_{zt}z)}$$

$$\overline{H}_2 = \mathrm{Re}\,\frac{\hat{E}_2}{\eta_2}e^{j(\omega t - k_{xt}x + k_{zt}z)}\overline{i}_y$$

$$\overline{E}_t = \mathrm{Re}\,\hat{E}_t[\cos\theta_i \overline{i}_x - \sin\theta_i \overline{i}_z]e^{j(\omega t - k_{xi}x - k_{zi}(z-d))}$$

$$\overline{H}_t = \mathrm{Re}\,\frac{\hat{E}_t}{\eta_1}e^{j(\omega t - k_{xi}x - k_{zi}(z-d))}\overline{i}_y$$

Boundary Conditions:

$$H_y(z=0_-) = H_y(z=0_+) \rightarrow \frac{\hat{E}_i + \hat{E}_r}{\eta_1} = \frac{\hat{E}_1 + \hat{E}_2}{\eta_2}$$

$$H_y(z=d_-) = H_y(z=d_+) \rightarrow \frac{\hat{E}_1 e^{-jk_{zt}d} + \hat{E}_2 e^{jk_{zt}d}}{\eta_2} = \frac{\hat{E}_t}{\eta_1}$$

$$E_x(z=0_-) = E_x(z=0_+) \rightarrow (\hat{E}_i - \hat{E}_r)\cos\theta_i = (\hat{E}_1 - \hat{E}_2)\cos\theta_t$$

$$E_x(z=d_-) = E_x(z=d_+) \rightarrow (\hat{E}_1 e^{-jk_{zt}d} - \hat{E}_2 e^{jk_{zt}d})\cos\theta_t = \hat{E}_t\cos\theta_i$$

$$k_1\sin\theta_i = k_2\sin\theta_t \rightarrow \sqrt{\varepsilon_1\mu_1}\,\sin\theta_i = \sqrt{\varepsilon_2\mu_2}\,\sin\theta_t$$

$$\begin{bmatrix} -1 & \frac{\eta_1}{\eta_2} & \frac{\eta_1}{\eta_2} & 0 \\ 0 & e^{-jk_{zt}d} & e^{jk_{zt}d} & -\frac{\eta_2}{\eta_1} \\ 1 & \frac{\cos\theta_t}{\cos\theta_i} & -\frac{\cos\theta_t}{\cos\theta_i} & 0 \\ 0 & e^{-jk_{zt}d} & -e^{jk_{zt}d} & -\frac{\cos\theta_i}{\cos\theta_t} \end{bmatrix} \begin{bmatrix} \hat{E}_r \\ \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_t \end{bmatrix} = \begin{bmatrix} \hat{E}_i \\ 0 \\ \hat{E}_i \\ 0 \end{bmatrix}$$

$$D = \det \begin{bmatrix} -1 & \frac{\eta_1}{\eta_2} & \frac{\eta_1}{\eta_2} & 0 \\ 0 & e^{-jk_{zt}d} & e^{jk_{zt}d} & -\frac{\eta_2}{\eta_1} \\ 1 & \frac{\cos\theta_t}{\cos\theta_i} & -\frac{\cos\theta_t}{\cos\theta_i} & 0 \\ 0 & e^{-jk_{zt}d} & -e^{jk_{zt}d} & -\frac{\cos\theta_i}{\cos\theta_t} \end{bmatrix}$$

$$= -4\cos k_{zt}d - 2j\sin k_{zt}d \left[\frac{\eta_1}{\eta_2}\frac{\cos\theta_i}{\cos\theta_t} + \frac{\eta_2}{\eta_1}\frac{\cos\theta_t}{\cos\theta_i}\right]$$

$$\hat{E}_r = \frac{1}{D} \det \begin{bmatrix} \hat{E}_i & \frac{\eta_1}{\eta_2} & \frac{\eta_1}{\eta_2} & 0 \\ 0 & e^{-jk_{zt}d} & e^{jk_{zt}d} & -\frac{\eta_2}{\eta_1} \\ \hat{E}_i & \frac{\cos\theta_t}{\cos\theta_i} & -\frac{\cos\theta_t}{\cos\theta_i} & 0 \\ 0 & e^{-jk_{zt}d} & -e^{jk_{zt}d} & -\frac{\cos\theta_i}{\cos\theta_t} \end{bmatrix}$$

$$= \frac{2j\hat{E}_i}{D} \sin k_{zt} d \left[\frac{\eta_2}{\eta_1} \frac{\cos\theta_t}{\cos\theta_i} - \frac{\eta_1}{\eta_2} \frac{\cos\theta_i}{\cos\theta_t} \right]$$

$$\hat{E}_1 = \frac{1}{D} \det \begin{bmatrix} -1 & \hat{E}_i & \frac{\eta_1}{\eta_2} & 0 \\ 0 & 0 & e^{jk_{zt}d} & -\frac{\eta_2}{\eta_1} \\ 1 & \hat{E}_i & -\frac{\cos\theta_t}{\cos\theta_i} & 0 \\ 0 & 0 & -e^{jk_{zt}d} & -\frac{\cos\theta_i}{\cos\theta_t} \end{bmatrix} = \frac{-2\hat{E}_i e^{jk_{zt}d}}{D} \left[\frac{\cos\theta_i}{\cos\theta_t} + \frac{\eta_2}{\eta_1} \right]$$

$$\hat{E}_2 = \frac{1}{D} \det \begin{bmatrix} -1 & \frac{\eta_1}{\eta_2} & \hat{E}_i & 0 \\ 0 & e^{-jk_{zt}d} & 0 & -\frac{\eta_2}{\eta_1} \\ 1 & \frac{\cos\theta_t}{\cos\theta_i} & \hat{E}_i & 0 \\ 0 & e^{-jk_{zt}d} & 0 & -\frac{\cos\theta_i}{\cos\theta_t} \end{bmatrix} = \frac{2\hat{E}_i e^{-jk_{zt}d}}{D} \left[\frac{\cos\theta_i}{\cos\theta_t} - \frac{\eta_2}{\eta_1} \right]$$

$$\hat{E}_t = \frac{1}{D} \det \begin{bmatrix} -1 & \frac{\eta_1}{\eta_2} & \frac{\eta_1}{\eta_2} & \hat{E}_i \\ 0 & e^{-jk_{zt}d} & e^{jk_{zt}d} & 0 \\ 1 & \frac{\cos\theta_t}{\cos\theta_i} & -\frac{\cos\theta_t}{\cos\theta_i} & \hat{E}_i \\ 0 & e^{-jk_{zt}d} & -e^{jk_{zt}d} & 0 \end{bmatrix} = \frac{-4\hat{E}_i}{D}$$

Check: Special Case, $\eta_1 = \eta_2$, $\theta_t = \theta_i$

$$D = -4e^{jk_{zt}d}$$

$$\hat{E}_r = 0,\ \hat{E}_1 = 1,\ \hat{E}_2 = 0,\ \hat{E}_t = \hat{E}_i e^{-jk_{zt}d}$$

b) Uniform plane waves in middle region $\rightarrow k_{zt}$ real $\rightarrow \cos\theta_t$ real

$$\rightarrow \sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}}\sin\theta_i < 1 \rightarrow \sin\theta_i < \sqrt{\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}}$$

c) $$\langle \overline{S}\rangle_t = \frac{1}{2}\,\mathrm{Re}\,\hat{\overline{E}}_t \times \hat{\overline{H}}^*_t$$

$$= \frac{1}{2}\,\mathrm{Re}\,\frac{|\hat{E}_t|^2}{\eta_1}[\cos\theta_i \overline{i}_z + \sin\theta_i \overline{i}_x]$$

Uniform plane wave in middle region [$\cos\theta_t$ real]

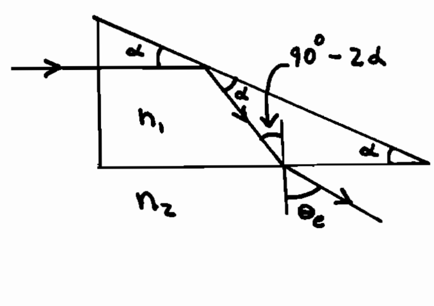
$$|\hat{E}_t|^2 = \frac{4|\hat{E}_i|^2}{4\cos^2 k_{zt}d + \sin^2 k_{zt}d\left[\frac{\eta_1}{\eta_2}\frac{\cos\theta_i}{\cos\theta_t} + \frac{\eta_2}{\eta_1}\frac{\cos\theta_t}{\cos\theta_i}\right]^2}$$

Non-uniform plane wave in middle region [$\cos\theta_t = -j|\cos\theta_t|$]

$$|\hat{E}_t|^2 = \frac{4|\hat{E}_i|^2}{\left\{2\cos k_{zt}d - \sin k_{zt}d\left[\frac{\eta_1}{\eta_2}\frac{\cos\theta_i}{|\cos\theta_t|} + \frac{\eta_2}{\eta_1}\frac{|\cos\theta_t|}{\cos\theta_i}\right]\right\}^2}$$

Oppositely traveling non-uniform plane waves in middle region allow time average power flow although each non-uniform plane wave alone cannot carry time average power.

Section 7.10



30. a) and b) $\sin\theta_c = \frac{n_2}{n_1}$, $\sin\theta_e = \frac{n_1}{n_2}\cos 2\alpha = \frac{\cos 2\alpha}{\sin\theta_c}$

$$\sin\theta_e = \frac{n_1}{n_2}\sin(90° - 2\alpha)$$

α	$\theta_c = 90° - \alpha$	n_2	n_1	θ_e
45°	45°	1	1.414	0
		1.33	1.88	
30°	60°	1	1.155	35.27°
		1.33	1.536	
60°	30°	1	2	-90°
		1.33	2.66	

31. $\sin\theta_t = 1.33 \sin\theta_i$

$$= (1.33)(.5) = .665 \rightarrow \theta_t = 41.69°$$

32. a) $\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1$

$$\sin\theta_3 = \frac{n_2}{n_3}\sin\theta_2 = \frac{n_1}{n_3}\sin\theta_1$$

b) $\sin\theta_{1c} = \frac{n_3}{n_1}$

c) Same critical angle with or without coating.

d) $\sin\theta_1 > \frac{n_2}{n_1}$ [Coating must have smaller index of refraction than light pipe.]

33. a) $\sin\theta_t = \frac{1}{n}\sin\theta = \frac{x}{nR}$

$$\frac{y - R}{\sin\theta_t} = \frac{R}{\sin(\theta - \theta_t)} \rightarrow y = R\left[\frac{\sin\theta_t}{\sin(\theta - \theta_t)} + 1\right]$$

$$\sin(\theta - \theta_t) = \sin\theta\cos\theta_t - \sin\theta_t\cos\theta$$

$$= \sin\theta\sqrt{1 - \sin^2\theta_t} - \sin\theta_t\sqrt{1 - \sin^2\theta}$$

$$= \frac{x}{R}\sqrt{1 - (\frac{x}{nR})^2} - \frac{x}{nR}\sqrt{1 - (\frac{x}{R})^2}$$

$$y = R\left[\frac{1}{\sqrt{n^2 - (\frac{x}{R})^2} - \sqrt{1 - (\frac{x}{R})^2}} + 1\right]$$

$$y<2R \rightarrow \sqrt{n^2 - (\frac{x}{R})^2} - \sqrt{1 - (\frac{x}{R})^2} > 1$$

$$n^2 - (\frac{x}{R})^2 > 1 + (1 - (\frac{x}{R})^2) + 2\sqrt{1 - (\frac{x}{R})^2}$$

$$\sqrt{1 - (\frac{x}{R})^2} < \frac{n^2 - 2}{2}$$

$$n^2 > 2 \rightarrow 1 - (\frac{x}{R})^2 < \frac{(n^2 - 2)^2}{4} \rightarrow (\frac{x}{R})^2 > 1 - \frac{(n^2 - 2)^2}{4} \qquad ; \sqrt{2} \le n \le 2$$

b) $$\sin\theta_t = \frac{\sin\theta}{n} = \frac{\alpha}{n}$$

$$\frac{\sin\theta_t}{R'} = \frac{\sin(90^\circ + \theta - \theta_t)}{R} = \frac{\cos(\theta - \theta_t)}{R}$$

$$R' = \frac{R\sin\theta_t}{\cos(\theta - \theta_t)} = \frac{R\alpha}{n[\cos\theta\cos\theta_t + \sin\theta\sin\theta_t]}$$

$$= \frac{\alpha R}{n[\sqrt{1 - \alpha^2}\sqrt{1 - (\frac{\alpha}{n})^2} + \frac{\alpha^2}{n}]}$$

$$= \frac{\alpha R}{[\sqrt{n^2 - (n\alpha)^2}\sqrt{n^2 - \alpha^2} + \alpha^2]}$$

CHAPTER 8

GUIDED ELECTROMAGNETIC WAVES

Section 8.1

1. From (2.6.4c)

$$C = \begin{cases} \dfrac{2\pi\varepsilon_o}{\cosh^{-1}(\frac{s}{a} + 1)} & \text{wire-plane} \\[2ex] \dfrac{2\pi\varepsilon_o}{\cosh^{-1}[\frac{D^2}{2a^2} - 1]} & \text{2 wire line} \end{cases}$$

$$L = \frac{\varepsilon_o \mu_o}{C} = \begin{cases} \dfrac{\mu_o}{2\pi} \cosh^{-1}(\frac{s}{a} + 1) & \text{wire-plane} \\[2ex] \dfrac{\mu_o}{2\pi} \cosh^{-1}[\frac{D^2}{2a^2} - 1] & \text{2 wire line} \end{cases}$$

2. a) $\dfrac{\partial v}{\partial z} = -L(z) \dfrac{\partial i}{\partial t}$

$$\frac{\partial i}{\partial z} = -C(z) \frac{\partial v}{\partial t}$$

$$\frac{\partial^2 v}{\partial z \partial t} = -L(z) \frac{\partial^2 i}{\partial t^2} = - \frac{\partial}{\partial z} \left[\frac{1}{C(z)} \frac{\partial i}{\partial z}\right] \rightarrow \frac{\partial^2 i}{\partial t^2} = \frac{1}{L(z)} \frac{\partial}{\partial z} \left[\frac{1}{C(z)} \frac{\partial i}{\partial z}\right]$$

$$\frac{\partial^2 i}{\partial z \partial t} = -C(z) \frac{\partial^2 v}{\partial t^2} = - \frac{\partial}{\partial z} \left[\frac{1}{L(z)} \frac{\partial v}{\partial z}\right] \rightarrow \frac{\partial^2 v}{\partial t^2} = \frac{1}{C(z)} \frac{\partial}{\partial z} \left[\frac{1}{L(z)} \frac{\partial v}{\partial z}\right]$$

b) $L(z) = L_o e^{\alpha z}$, $C(z) = C_o e^{-\alpha z}$

$$v(z,t) = \mathrm{Re}\,\hat{v}(z) e^{j\omega t}, \quad i(z,t) = \mathrm{Re}\,\hat{i}(z) e^{j\omega t}$$

$$\frac{\partial^2 i}{\partial t^2} - \frac{1}{L(z)C(z)}\frac{\partial^2 i}{\partial z^2} + \frac{1}{L(z)C^2(z)}\frac{\partial i}{\partial z}\frac{dC(z)}{dz} = 0$$

$$\frac{\partial^2 v}{\partial t^2} - \frac{1}{L(z)C(z)}\frac{\partial^2 v}{\partial z^2} + \frac{1}{L^2(z)C(z)}\frac{\partial v}{\partial z}\frac{dL(z)}{dz} = 0$$

$$\frac{\partial^2 i}{\partial t^2} - \frac{1}{L_oC_o}\frac{\partial^2 i}{\partial z^2} - \frac{\alpha}{L_oC_o}\frac{\partial i}{\partial z} = 0$$

$$\frac{\partial^2 v}{\partial t^2} - \frac{1}{L_oC_o}\frac{\partial^2 v}{\partial z^2} + \frac{\alpha}{L_oC_o}\frac{\partial v}{\partial z} = 0$$

$$\frac{d^2\hat{i}(z)}{dz^2} + \alpha\frac{d\hat{i}(z)}{dz} + L_oC_o\omega^2\hat{i}(z) = 0$$

$$\hat{i}(z) = \hat{I}e^{pz} \rightarrow p^2 + \alpha p + L_oC_o\omega^2 = 0$$

$$p = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - L_oC_o\omega^2} = \frac{\alpha}{2} \pm j\beta; \quad \beta = \sqrt{L_oC_o\omega^2 - \frac{\alpha^2}{4}}$$

$$\hat{i}(z) = [I_1e^{j\beta z} + I_2e^{-j\beta z}]e^{-\frac{\alpha}{2}z}$$

$$\hat{v}(z) = -\frac{1}{j\omega C(z)}\frac{d\hat{i}}{dz} = -\frac{1}{j\omega C_o}[I_1(j\beta - \frac{\alpha}{2})e^{j\beta z} - I_2(j\beta + \frac{\alpha}{2})e^{-j\beta z}]\, e^{\frac{\alpha}{2}z}$$

c) $$\hat{v}(z{=}0) = V_o = -\frac{1}{j\omega C_o}[I_1(j\beta - \frac{\alpha}{2}) - I_2(j\beta + \frac{\alpha}{2})]$$

Open Circuited End: $\hat{i}(z{=}\ell) = 0 \rightarrow I_1e^{j\beta\ell} + I_2e^{-j\beta\ell} = 0$

$$I_1 = \frac{-j\omega C_oV_oe^{-j\beta\ell}}{[(j\beta+\frac{\alpha}{2})e^{j\beta\ell} + (j\beta-\frac{\alpha}{2})e^{-j\beta\ell}]}\ ; \quad I_2 = \frac{j\omega C_oV_oe^{j\beta\ell}}{[(j\beta+\frac{\alpha}{2})e^{j\beta\ell} + (j\beta-\frac{\alpha}{2})e^{-j\beta\ell}]}$$

$$\hat{i}(z) = \frac{-j\omega C_oV_o}{[(j\beta+\frac{\alpha}{2})e^{j\beta\ell} + (j\beta-\frac{\alpha}{2})e^{-j\beta\ell}]}[e^{j\beta(z-\ell)} - e^{-j\beta(z-\ell)}]e^{-\frac{\alpha}{2}z}$$

$$= \frac{2\omega C_oV_o\sin\beta(z-\ell)e^{-\frac{\alpha}{2}z}}{[(j\beta+\frac{\alpha}{2})e^{j\beta\ell} + (j\beta-\frac{\alpha}{2})^{-j\beta\ell}]}$$

$$\hat{v}(z) = -\frac{1}{j\omega C(z)}\frac{d\hat{i}}{dz} = \frac{2jV_o[\beta\cos\beta(z-\ell) - \frac{\alpha}{2}\sin\beta(z-\ell)]e^{\frac{\alpha}{2}z}}{[(j\beta+\frac{\alpha}{2})e^{j\beta\ell} + (j\beta-\frac{\alpha}{2})e^{-j\beta\ell}]}$$

Short Circuited End: $\hat{v}(z=\ell) = 0 \rightarrow I_1(j\beta - \frac{\alpha}{2})e^{j\beta\ell} - I_2(j\beta + \frac{\alpha}{2})e^{-j\beta\ell} = 0$

$$I_1 = \frac{\omega C_o V_o e^{-j\beta\ell}}{2(j\beta - \frac{\alpha}{2})\sin\beta\ell}, \quad I_2 = \frac{\omega C_o V_o e^{j\beta\ell}}{2(j\beta + \frac{\alpha}{2})\sin\beta\ell}$$

$$\hat{v}(z) = \frac{jV_o}{2\sin\beta\ell}[e^{j\beta(z-\ell)} - e^{-j\beta(z-\ell)}]e^{\frac{\alpha}{2}z}$$

$$= -\frac{V_o\sin\beta(z-\ell)e^{\frac{\alpha}{2}z}}{\sin\beta\ell}$$

$$\hat{i}(z) = -\frac{1}{j\omega L(z)}\frac{d\hat{v}}{dz} = \frac{V_o}{jL_o\omega\sin\beta\ell}[\beta\cos\beta(z-\ell) + \frac{\alpha}{2}\sin\beta(z-\ell)]e^{-\frac{\alpha}{2}z}$$

d) Waves decay if $\frac{\alpha^2}{4} > \omega^2 L_o C_o$

$$\text{Cut off} \rightarrow \omega_{co} = \frac{\alpha}{2\sqrt{L_o C_o}}$$

e) Short circuited line resonance $\rightarrow \sin\beta\ell = 0 \rightarrow \beta\ell = \ell\sqrt{L_o C_o\omega^2 - \frac{\alpha^2}{4}} = n\pi \quad n=1,2,3\ldots$

$$\omega_n = \left[\frac{(\frac{n\pi}{\ell})^2 + \frac{\alpha^2}{4}}{L_o C_o}\right]^{1/2}$$

f) Open circuited line resonance $\rightarrow j\beta(e^{j\beta\ell} + e^{-j\beta\ell}) + \frac{\alpha}{2}(e^{j\beta\ell} - e^{-j\beta\ell}) = 0$

$$\beta\cos\beta\ell + \frac{\alpha}{2}\sin\beta\ell = 0 \rightarrow \tan\beta\ell = -\frac{2\beta}{\alpha}$$

3. a) $\overline{E} = \frac{v(t)}{\alpha r}\overline{i}_\phi, \; \sigma_f = \varepsilon E_\phi = \frac{\varepsilon v(t)}{\alpha r}, \; q_t = \int_a^b \sigma_f D dr = \frac{\varepsilon v(t) D}{\alpha}\ln\frac{b}{a}$

$$\overline{H} = H_r(r)\overline{i}_r \rightarrow \Phi = \mu H_r(r) r\alpha D \rightarrow H_r(r) = \frac{\Phi}{\mu\alpha D r}$$

$$\int_a^b H_r(r)dr = \frac{\Phi}{\mu\alpha D}\ln\frac{b}{a} = i \rightarrow H_r(r) = \frac{i}{r\ln\frac{b}{a}}$$

b) $C = \dfrac{q_T}{Dv(t)} = \dfrac{\varepsilon \ln \frac{b}{a}}{\alpha}, \quad L = \dfrac{\Phi}{Di} = \dfrac{\mu\alpha}{\ln \frac{b}{a}}$

$LC = \varepsilon\mu, \quad Z_o = \sqrt{\dfrac{L}{C}} = \sqrt{\dfrac{\mu}{\varepsilon}}\,\dfrac{\alpha}{\ln \frac{b}{a}}$

4. a) $\nabla \times \overline{E} = -\mu \dfrac{\partial \overline{H}}{\partial t} \rightarrow \dfrac{\partial E_x}{\partial z} = -\mu \dfrac{\partial H_y}{\partial t}$

$\nabla \times \overline{H} = \overline{J}_f + \varepsilon \dfrac{\partial \overline{E}}{\partial t} \rightarrow -\dfrac{\partial H_y}{\partial z} = J_x + \varepsilon \dfrac{\partial E_x}{\partial t}$

$-\dfrac{\partial^2 H_y}{\partial z \partial t} = \dfrac{\partial J_x}{\partial t} + \varepsilon \dfrac{\partial^2 E_x}{\partial t^2} = \dfrac{1}{\mu} \dfrac{\partial^2 E_x}{\partial z^2} \rightarrow \omega_p^2 E_x + \dfrac{\partial^2 E_x}{\partial t^2} - c^2 \dfrac{\partial^2 E_x}{\partial z^2} = 0$

b) $E_x(z,t) = \dfrac{v(z,t)}{s} \rightarrow \dfrac{\partial^2 v}{\partial t^2} + \omega_p^2 v - c^2 \dfrac{\partial^2 v}{\partial z^2} = 0$

$H_y(z,t) = \dfrac{i(z,t)}{D}$

c) $v(z,t) = \mathrm{Re}\,\hat{v}e^{j(\omega t - kz)} \rightarrow -\omega^2 + \omega_p^2 + k^2c^2 = 0$

$$k = \pm \frac{\sqrt{\omega^2 - \omega_p^2}}{c}$$

k real (propagation) $\rightarrow \omega > \omega_p$

k imaginary (evanescence) $\omega < \omega_p$

d) $\hat{v}(z) = V_1 \sin kz + V_2 \cos kz$

$\hat{v}(z=0) = 0 = V_2$

$\hat{v}(z=-\ell) = V_o = -V_1 \sin k\ell$

$\hat{v}(z) = -\dfrac{V_o \sin kz}{\sin k\ell}, \quad v(z,t) = \mathrm{Re}\,\hat{v}(z)e^{j\omega t} = -\dfrac{V_o \sin kz}{\sin k\ell} \cos\omega t$

$-\dfrac{\mu}{D}\dfrac{\partial i}{\partial t} = \dfrac{1}{s}\dfrac{\partial v}{\partial z} \rightarrow \hat{i}(z) = +\dfrac{D V_o k}{\mu s j\omega}\,\dfrac{\cos kz}{\sin k\ell}$

$$i(z,t) = \mathrm{Re}\,\hat{i}(z)e^{j\omega t} = \frac{V_o kD}{\omega\mu s}\frac{\cos kz}{\sin k\ell}\sin\omega t$$

e) Resonance $\rightarrow \sin k\ell = 0 \rightarrow k\ell = n\pi \rightarrow \omega^2 = \omega_p^2 + (\frac{n\pi c}{\ell})^2$

5. a) $i(z,t) = \frac{C}{\Delta z}\frac{\partial}{\partial t}[v(z-\Delta z) - v(z)]; \quad v(z,t) = \frac{L}{\Delta z}\frac{\partial}{\partial t}[i(z) - i(z+\Delta z)]$

$$\lim_{z\to 0} i(z,t) = -C\frac{\partial^2 v}{\partial t\partial z}; \quad v(z,t) = -L\frac{\partial^2 i}{\partial t\partial z}$$

b) $i(z,t) = \mathrm{Re}\,\hat{i}e^{j(\omega t - kz)}, \quad v(z,t) = \mathrm{Re}\,\hat{v}e^{j(\omega t - kz)}$

$$\hat{i} = -C\omega k\hat{v}; \quad \hat{v} = -L\omega k\hat{i}$$

$$\hat{i} = +LC\omega^2 k^2\hat{i} \rightarrow LC\omega^2 k^2 = 1 \rightarrow k = \frac{1}{\omega\sqrt{LC}}$$

c) $v_p = \frac{\omega}{k} = \omega^2\sqrt{LC}$

$$v_g = \frac{d\omega}{dk} = -\omega^2\sqrt{LC}$$

Such systems are called backward wave because the group velocity is opposite in direction to the phase velocity.

d) $\hat{v}(z) = V_1\sin kz + V_2\cos kz$

$$\hat{v}(z=0) = 0 = V_2$$

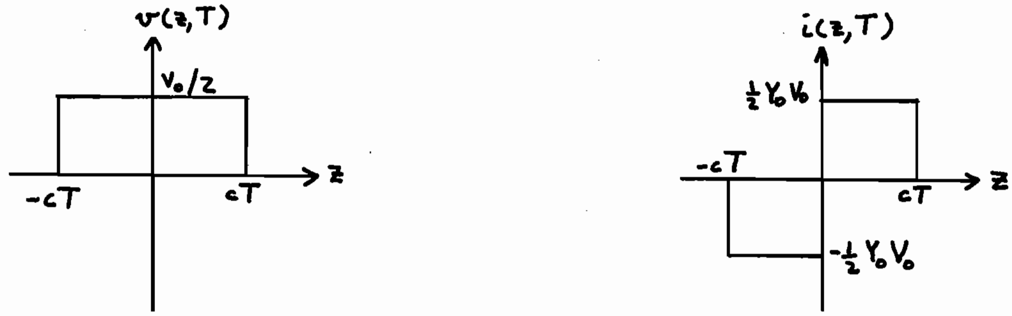
$$\hat{v}(z=-\ell) = V_o = -V_1\sin k\ell \rightarrow \hat{v}(z) = \frac{-V_o}{\sin k\ell}\sin kz$$

$$\hat{i}(z) = -Cj\omega\frac{d\hat{v}}{dz} = \frac{j\omega CV_o k\cos kz}{\sin k\ell} = j\sqrt{\frac{C}{L}}V_o\frac{\cos kz}{\sin k\ell}$$

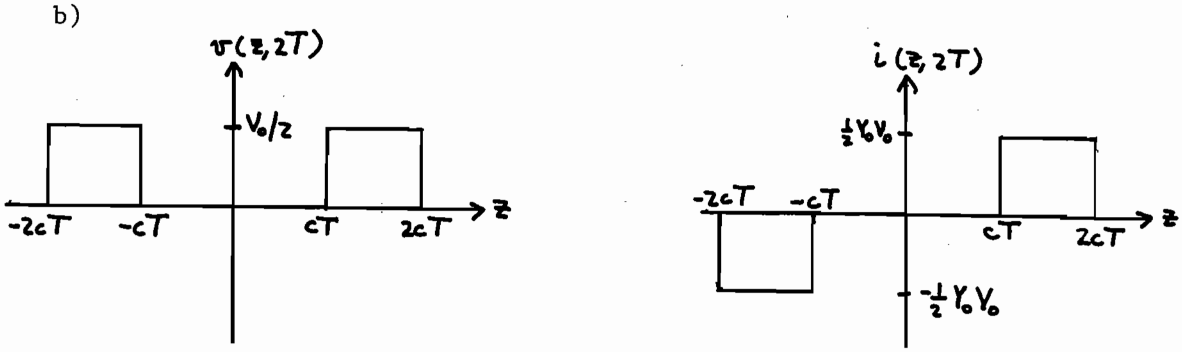
e) Resonance $\rightarrow \sin k\ell = 0 \rightarrow k\ell = n\pi \rightarrow \omega_n = \frac{1}{(\frac{n\pi}{\ell})\sqrt{LC}}$

Section 8.2

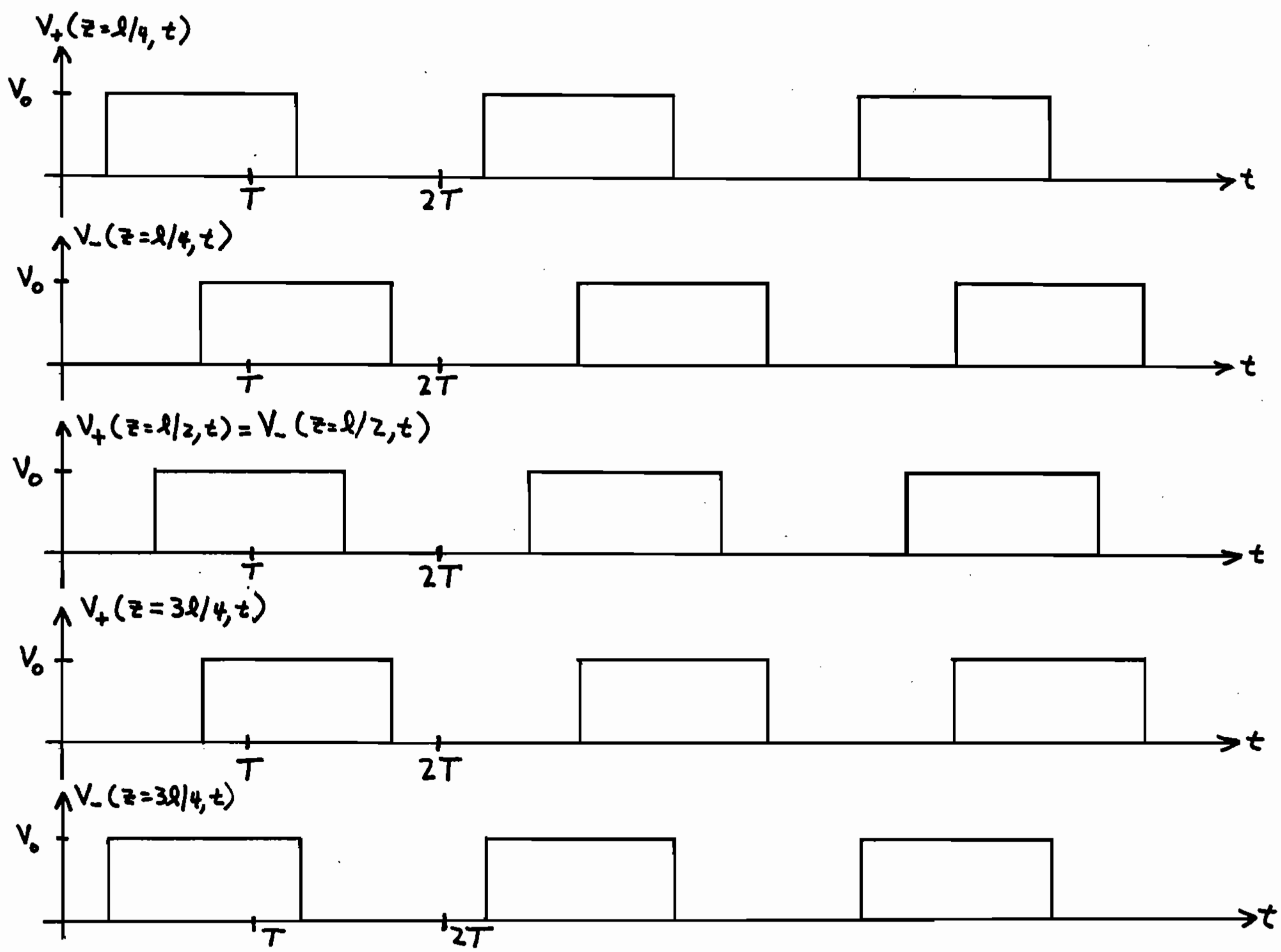
6. a)

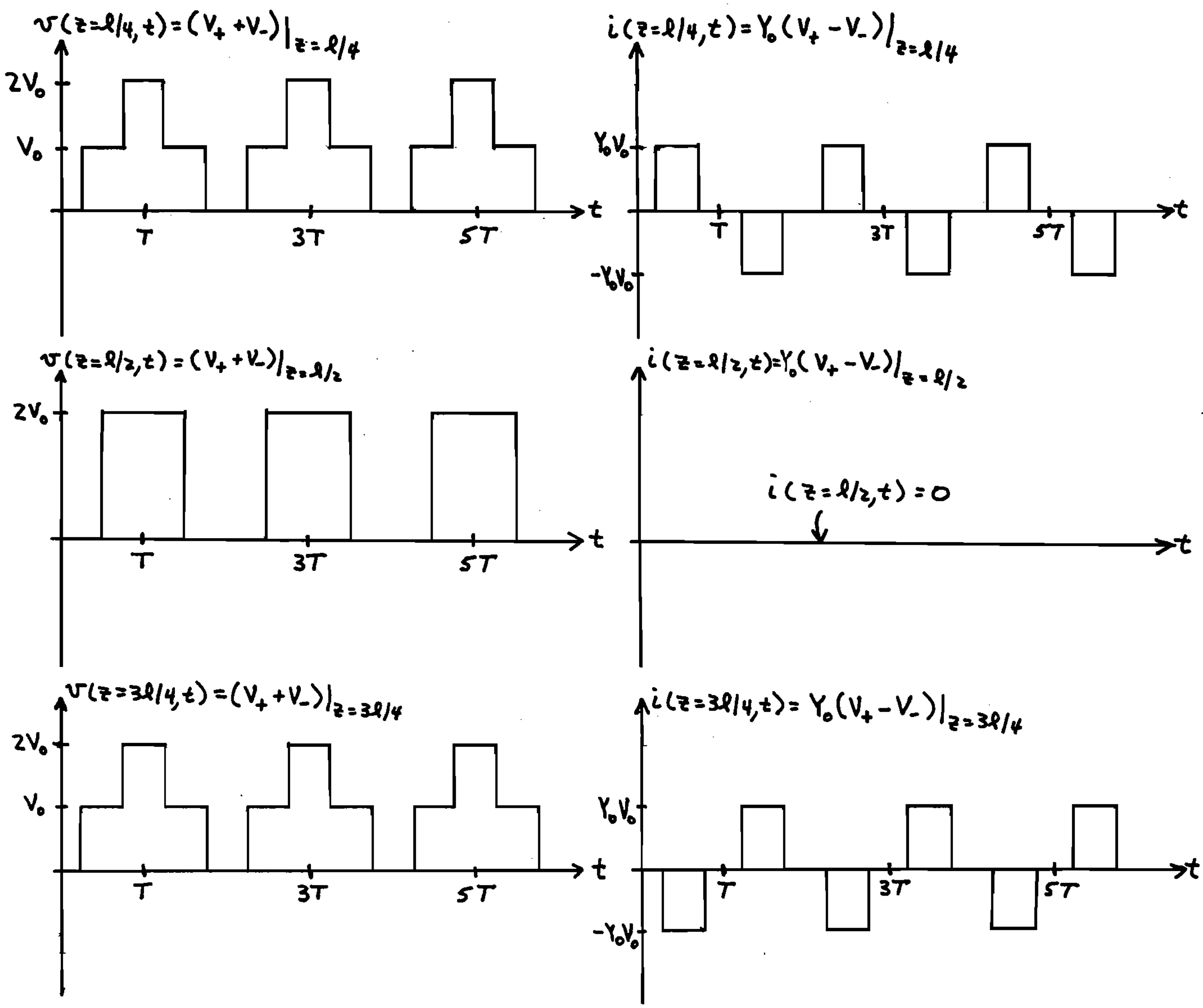


b)



7.





8. a) $v(t=0) = \dfrac{V_o R_L}{R_L + R_s} = V_+ + V_-$ $\qquad V_+ = \dfrac{V_o}{2}\dfrac{(R_L + Z_o)}{(R_L + R_s)}$

$\longrightarrow$

$i(t=0) = \dfrac{V_o}{R_L + R_s} = Y_o(V_+ - V_-)$ $\qquad V_- = \dfrac{V_o}{2}\dfrac{(R_L - Z_o)}{(R_L + R_s)}$

b) $V_{+n} = A(\Gamma_s\Gamma_L)^n;\ \Gamma_s = \dfrac{R_s - Z_o}{R_s + Z_o},\ \Gamma_L = \dfrac{R_L - Z_o}{R_L + Z_o}$

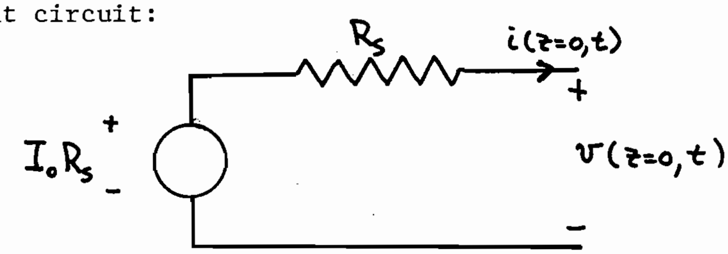
$V_{-n} = \Gamma_L V_{+n} = A\Gamma_L(\Gamma_s\Gamma_L)^n;$

$$V_{+n=0} = A = \frac{V_o}{2}\left(\frac{R_L + Z_o}{R_L + R_s}\right)$$

$$V_n = V_{+n} + V_{-n} = \frac{V_o}{2}\left(\frac{R_L + Z_o}{R_L + R_s}\right)\left[1 + \frac{R_L - Z_o}{R_L + Z_o}\right](\Gamma_s\Gamma_L)^n$$

$$= \frac{V_o R_L}{R_L + R_s}(\Gamma_s\Gamma_L)^n$$

9. Thevenin equivalent circuit:



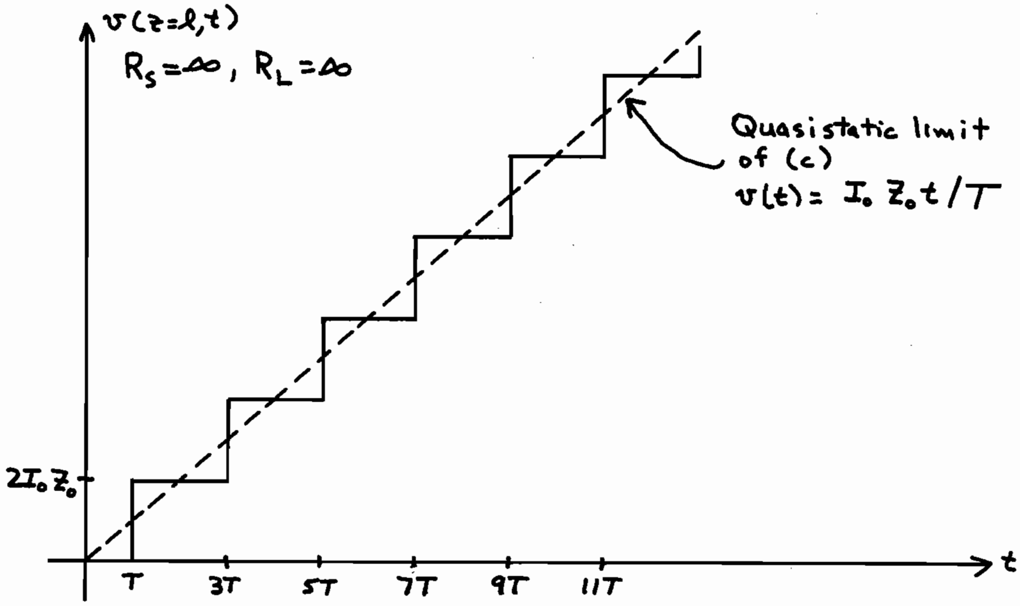
a) $$V_n = \frac{I_o R_L R_s}{R_L + R_s}[1 - (\Gamma_s\Gamma_L)^n]; \quad \Gamma_s = \frac{R_s - Z_o}{R_s + Z_o}, \quad \Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$$

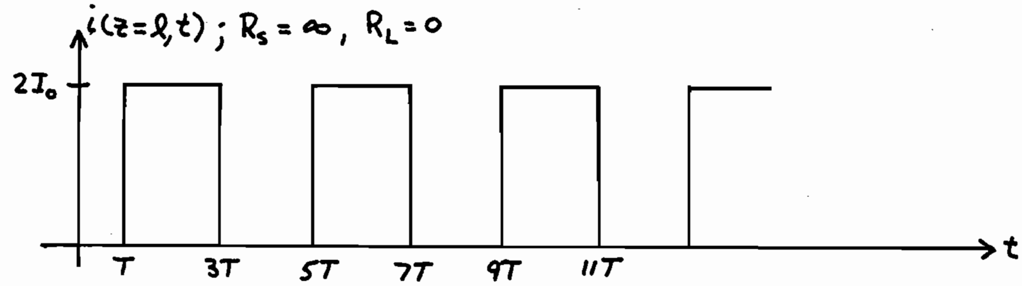
$$I_n = \frac{V_n}{R_L} = \frac{I_o R_s}{R_L + R_s}[1 - (\Gamma_s\Gamma_L)^n]$$

b) $$V_n = \frac{I_o R_s \Gamma_o (1 + \Gamma_L)[1 - (\Gamma_s\Gamma_L)^n]}{1 - \Gamma_s\Gamma_L}; \quad \Gamma_o = \frac{Z_o}{R_s + Z_o}$$

$$\lim_{\substack{R_L\to\infty \\ R_s\to\infty}} V_n = \lim_{\substack{\Gamma_s\to 1 \\ \Gamma_L\to 1}} V_n = 2I_o Z_o n$$

$$\lim_{\substack{R_L\to 0 \\ R_s\to\infty}} I_n = \lim_{\substack{\Gamma_s\to 1 \\ \Gamma_L\to -1}} I_n = I_o[1 - (-1)^n] = \begin{cases} 0 & n \text{ even} \\ 2I_o & n \text{ odd} \end{cases}$$



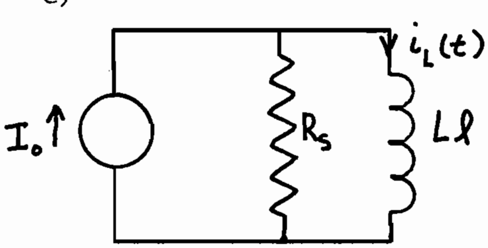


c) $I_o = C\ell \dfrac{dv}{dt} \rightarrow V = \dfrac{I_o}{C\ell} t = I_o Z_o \dfrac{t}{T}$; $T = \ell\sqrt{LC}$, $Z_o = \sqrt{\dfrac{L}{C}}$

d) $R_L = 0 \rightarrow \Gamma_L = -1$

$$I_n = I_o[1 - (-\Gamma_s)^n]; \quad \Gamma_s = \frac{R_s - Z_o}{R_s + Z_o} = \frac{R_s - \sqrt{\frac{L}{C}}}{R_s + \sqrt{\frac{L}{C}}}$$

e)



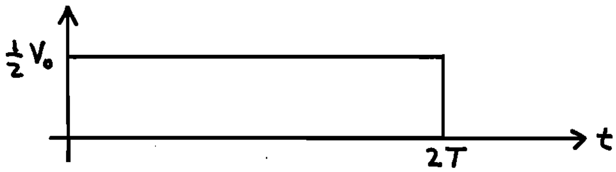
$$i_L(t) = I_o(1 - e^{-t/\tau}); \quad \tau = \frac{L\ell}{R}$$

$$\Gamma_s = \frac{\frac{R_s}{L\ell} - \frac{1}{\ell\sqrt{LC}}}{\frac{R_s}{L\ell} + \frac{1}{\ell\sqrt{LC}}} = \frac{\frac{1}{\tau} - \frac{1}{T}}{\frac{1}{\tau} + \frac{1}{T}} = \frac{T - \tau}{T + \tau}$$

d) and e) approximately equal if $\dfrac{T}{\tau} << 1$ $(T = \ell\sqrt{LC})$

10. a) $$\left.\begin{aligned} v(t=0) &= V_o = V_+ + V_- \\ i(t=0) &= 0 = Y_o(V_+ - V_-) \end{aligned}\right\} \longrightarrow V_+ = V_- = \frac{V_o}{2}$$

b) $v(z=\ell,t) = V_+(z=\ell,t);\ V_-(z=\ell,t) = 0$

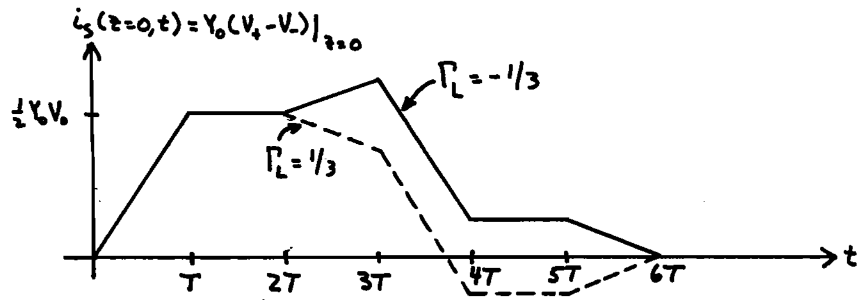


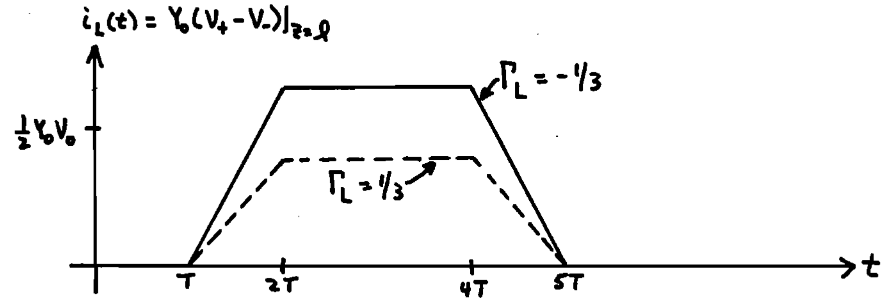
11. $$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$$

$$R_L = 2Z_o \rightarrow \Gamma_L = \frac{1}{3}$$

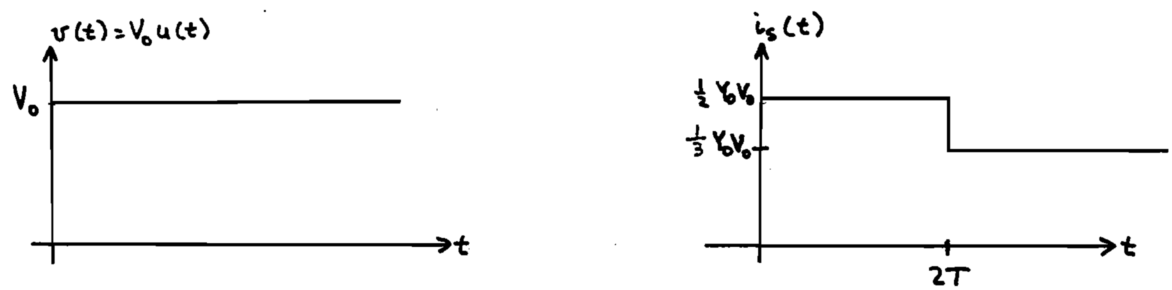
$$R_L = \frac{1}{2} Z_o \rightarrow \Gamma_L = -\frac{1}{3}$$

t	z = 0	z = ℓ
$0 < t < T$	$V_+ = \frac{V_o t}{2T},\ V_- = 0$	$V_+ = 0,\ V_- = 0$
$T < t < 2T$	$V_+ = \frac{V_o}{2},\ V_- = 0$	$V_+ = \frac{V_o(t-T)}{2T},\ V_- = \Gamma_L V_+$
$2T < t < 3T$	$V_+ = \frac{V_o}{2},\ V_- = \frac{\Gamma_L V_o(t-2T)}{2T}$	$V_+ = \frac{V_o}{2},\ V_- = \Gamma_L V_+$
$3T < t < 4T$	$V_+ = \frac{V_o(4T-t)}{2T},\ V_- = \Gamma_L \frac{V_o}{2}$	$V_+ = \frac{V_o}{2},\ V_- = \Gamma_L V_+$
$4T < t < 5T$	$V_+ = 0,\ V_- = \frac{\Gamma_L V_o}{2}$	$V_+ = \frac{V_o(5T-t)}{2T},\ V_- = \Gamma_L V_+$
$5T < t < 6T$	$V_+ = 0,\ V_- = \frac{V_o \Gamma_L (6T-t)}{2T}$	$V_+ = 0,\ V_- = 0$
$t > 6T$	$V_+ = 0,\ V_- = 0$	$V_+ = 0,\ V_- = 0$

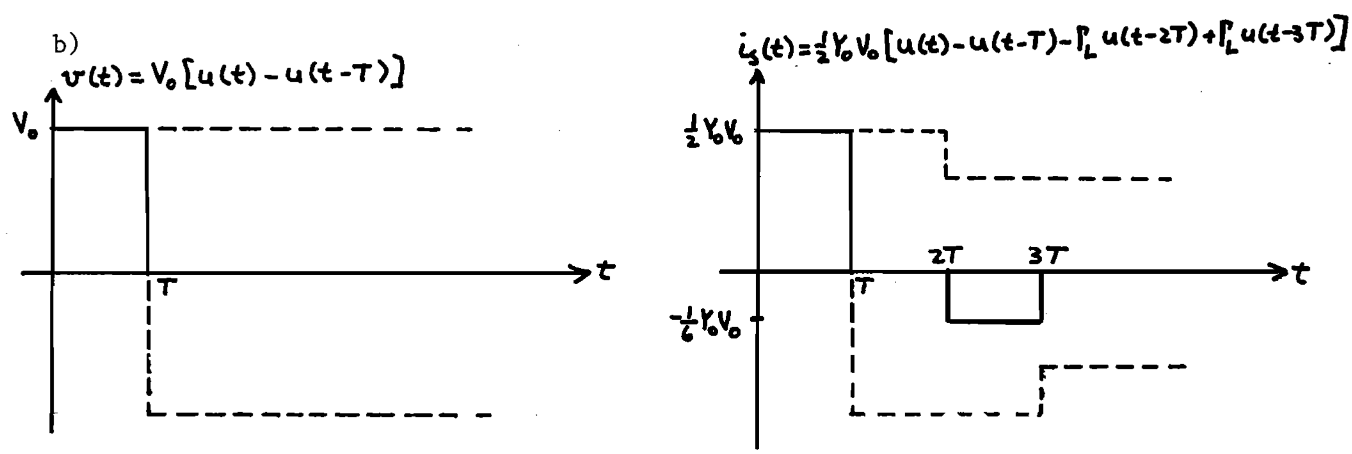


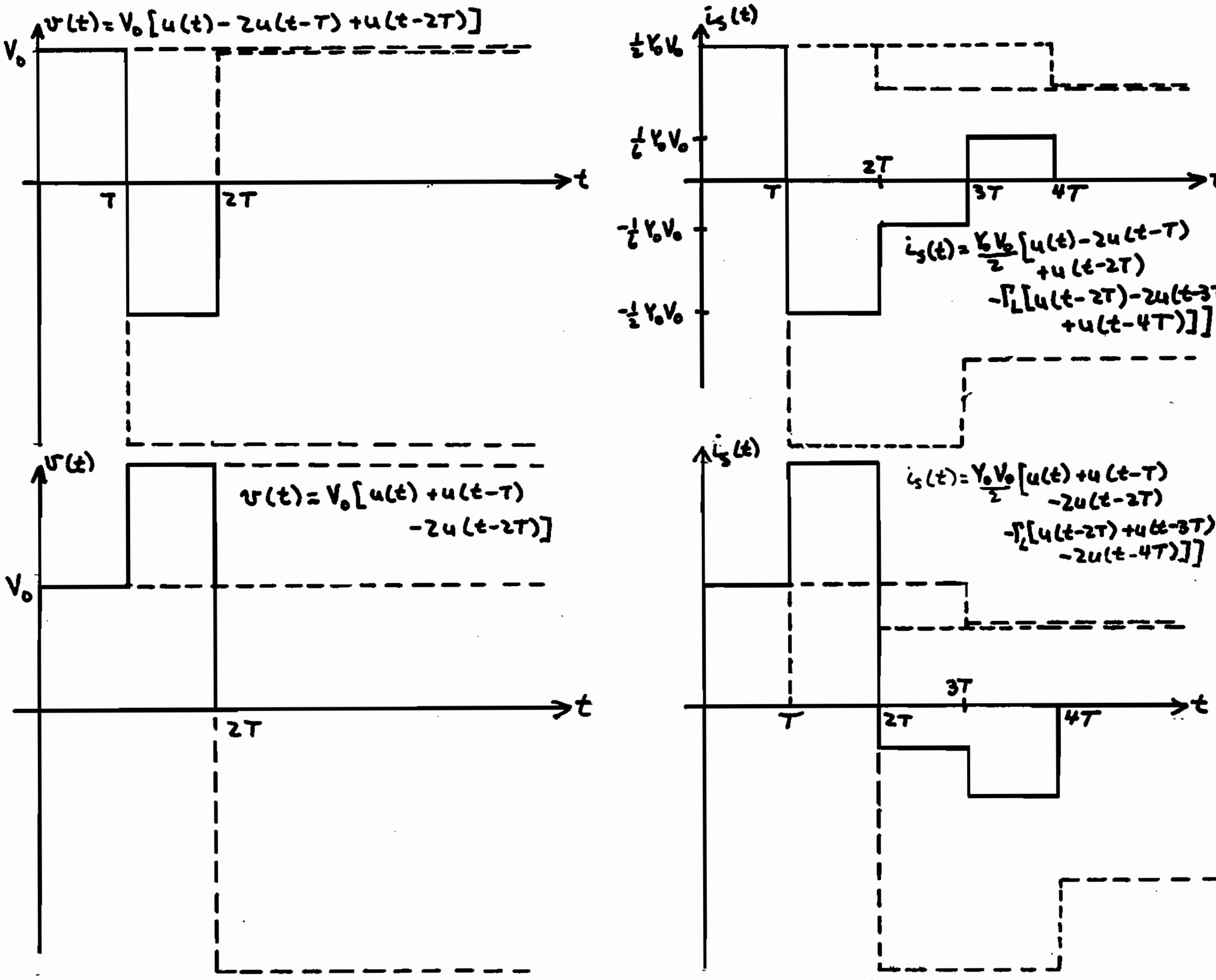


12. a) $v(t) = V_o u(t) \rightarrow i_s(t) = \frac{Y_o V_o}{2}\left[u(t) - \Gamma_L u(t-2T)\right]; \ \Gamma_L = \frac{1}{3}$

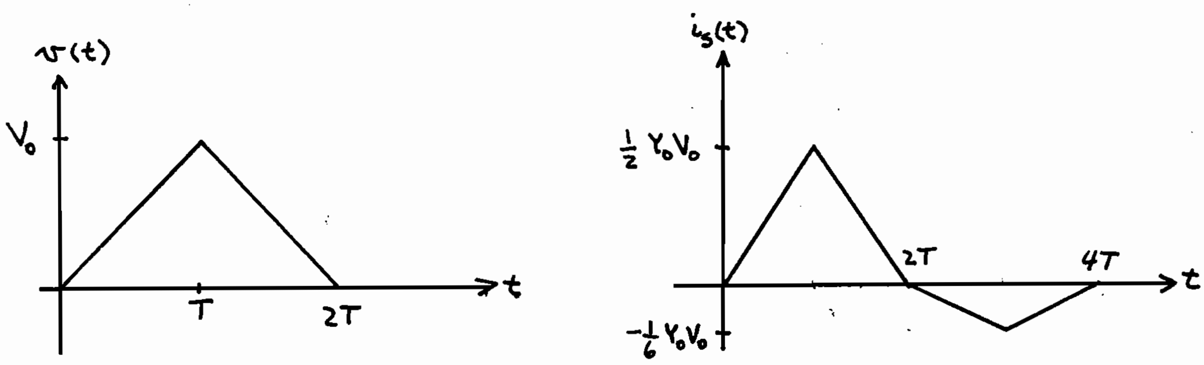


b)





c) Triangular voltage waveform is integral of $v(t) = V_o[u(t) - 2u(t-T) + u(t-2T)]$. The source current is then the integral of $i_s(t) = \frac{Y_oV_o}{2}[u(t) - 2u(t-T) + u(t-2T) - \Gamma_L[u(t-2T) - 2u(t-3T) + u(t-4T)]]$

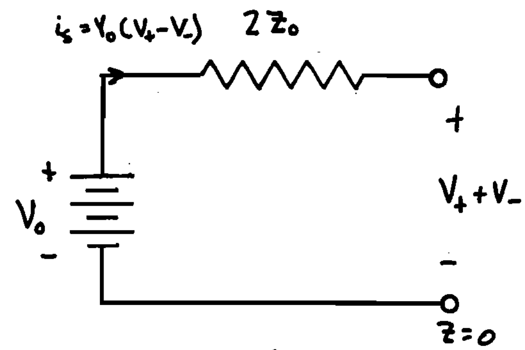


13. $v(t=0_-) = \frac{1}{2} V_o = V_+ + V_-$ $\rightarrow$ $V_+ = \frac{3}{8} V_o$

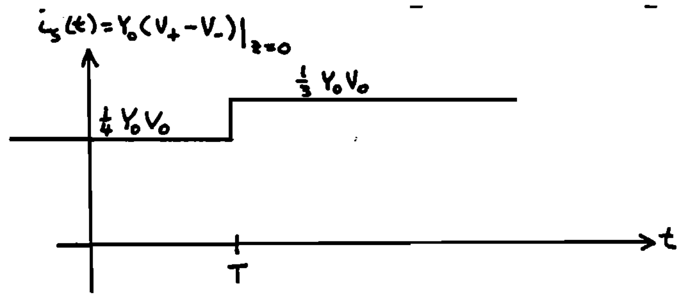
$i(t=0_-) = \frac{V_o Y_o}{4} = Y_o(V_+ - V_-)$ $V_- = \frac{1}{8} V_o$

a)

t	z = 0	z = ℓ
$0 < t < T$	$V_+ = \frac{3}{8} V_o$	$V_+ = \frac{3}{8} V_o$
	$V_- = \frac{1}{8} V_o$	$V_- = 0$
$T < t < 2T$	$V_+ = \frac{1}{3} V_o$	$V_+ = \frac{3}{8} V_o$
	$V_- = 0$	$V_- = 0$

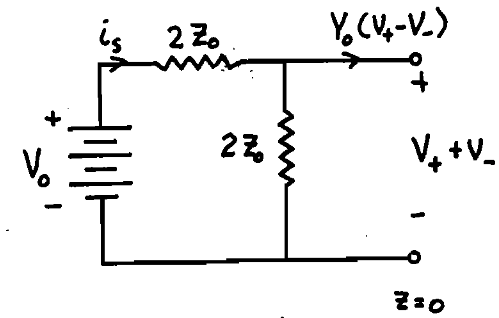


$V_+ = \frac{V_o + V_-}{3}$



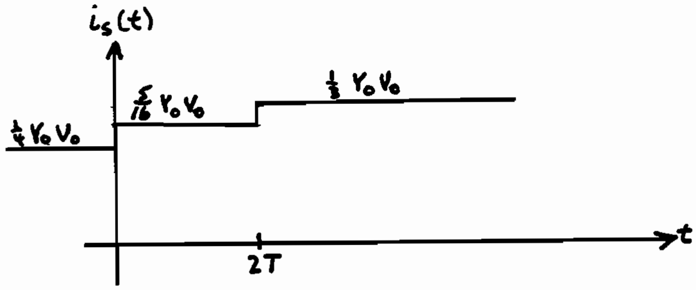
b)

t	z = 0	z = ℓ
$0 < t < T$	$V_+ = \frac{V_o}{4}$	$V_+ = \frac{3}{8} V_o$
	$V_- = \frac{1}{8} V_o$	$V_- = \frac{1}{8} V_o$
$T < t < 2T$	$V_+ = \frac{V_o}{4}$	$V_+ = \frac{V_o}{4}$
	$V_- = \frac{1}{8} V_o$	$V_- = \frac{V_o}{12}$
$2T < t < 3T$	$V_+ = \frac{V_o}{4}$	$V_+ = \frac{V_o}{4}$
	$V_- = \frac{V_o}{12}$	$V_- = \frac{V_o}{12}$



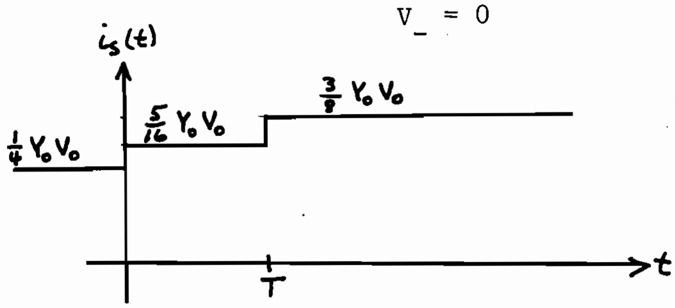
$V_+ = \frac{1}{4} V_o$

$i_s(t) = \frac{Y_o}{2}(3V_+ - V_-)$



c)

t	$z = 0$	$z = \ell$
$0 < t < T$	$V_+ = \frac{V_o}{4}$	$V_+ = \frac{3}{8} V_o$
	$V_- = \frac{1}{8} V_o$	$V_- = 0$
$T < t < 2T$	$V_+ = \frac{V_o}{4}$	$V_+ = \frac{V_o}{4}$
	$V_- = 0$	$V_- = 0$



14. a) Inductive Load

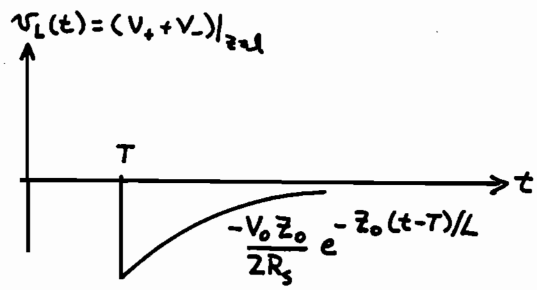
$$v(t=0_-) = 0 = V_+ + V_-$$
$$i(t=0_-) = \frac{V_o}{R_s} = Y_o(V_+ - V_-)$$
$$\rightarrow V_+ = -V_- = \frac{V_oZ_o}{2R_s}$$

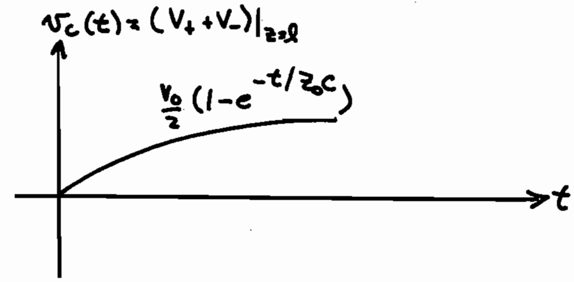
Capacitive Load

$$v(t=0_-) = 0 = V_+ + V_-$$
$$i(t=0_-) = \frac{V_o}{2Z_o} = Y_o(V_+ - V_-)$$
$$\rightarrow V_+ = -V_- = \frac{V_o}{4}$$

b)

t	$z = 0$	$z = \ell$	
$0 < t < T$	$V_+ = 0$	$V_+ = \frac{V_oZ_o}{2R_s}$	
	$V_- = \frac{-V_oZ_o}{2R_s}$	$V_- = \frac{-V_oZ_o}{2R_s}$	Inductive
$T < t < 2T$	$V_+ = 0$	$V_+ = 0$	
	$V_- = \frac{-V_oZ_o}{2R_s}$	$V_- = \frac{-V_oZ_o}{2R_s} e^{-Z_o(t-T)/L}$	



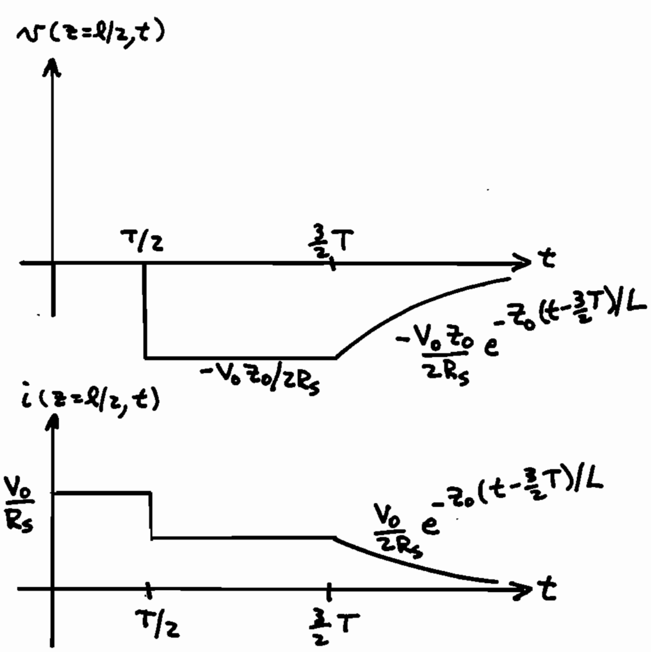


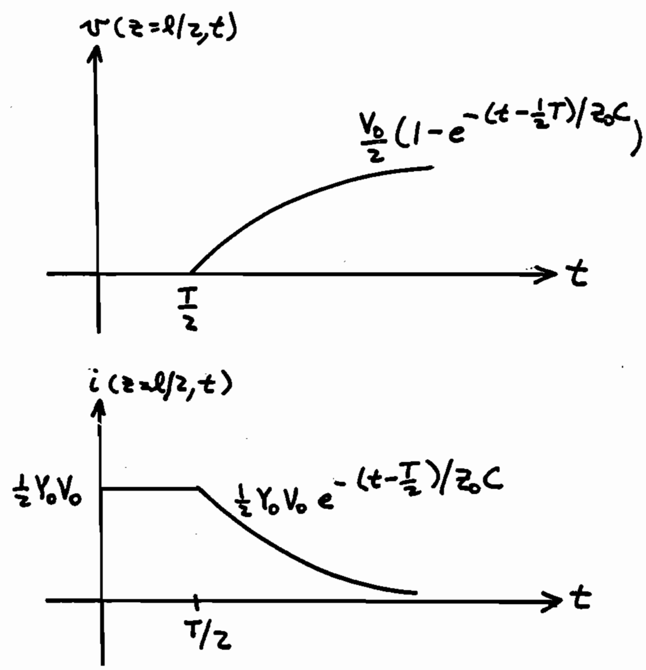
Capacitive

t	z = 0	z = ℓ
$0 < t < T$	$V_+ = \frac{V_o}{4}$	$V_+ = \frac{V_o}{4}$
	$V_- = \frac{-V_o}{4}$	$V_- = \frac{V_o}{2}(\frac{1}{2} - e^{-t/Z_oC})$
$T < t < 2T$	$V_+ = \frac{V_o}{4}$	$V_+ = \frac{V_o}{4}$
	$V_- = \frac{V_o}{2}(\frac{1}{2} - e^{-(t-T)/Z_oC})$	$V_- = \frac{V_o}{2}(\frac{1}{2} - e^{-t/Z_oC})$

Inductive Load

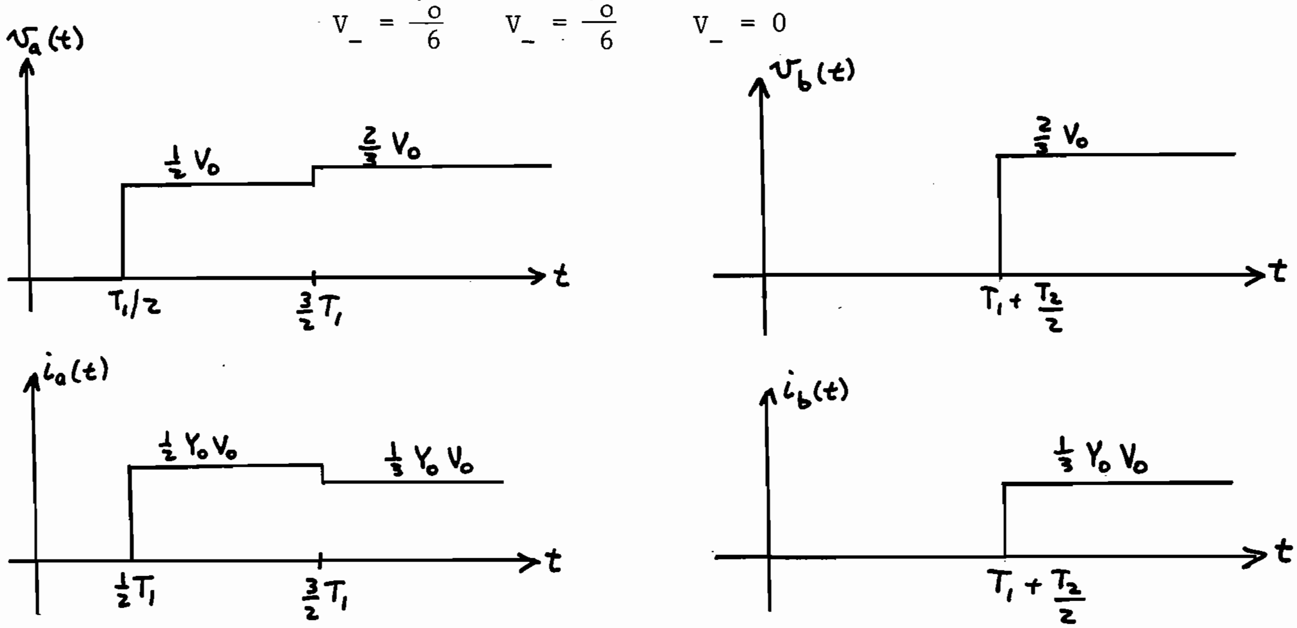
Capacitive Load





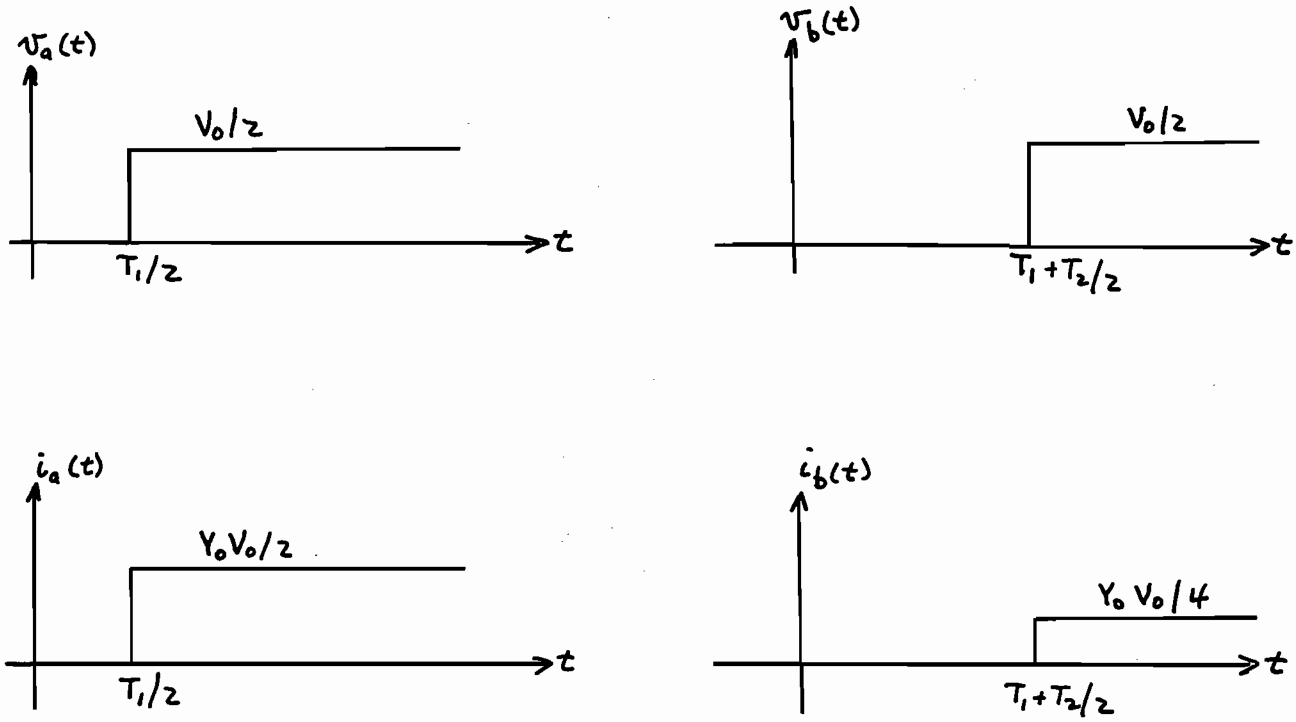
15. a)

t	$z = 0$	$z = \ell_{1-}$	$z = \ell_{1+}$
$0 < t < T_1$	$V_+ = \frac{V_o}{2}$	$V_+ = 0$	$V_+ = 0$
	$V_- = 0$	$V_- = 0$	$V_- = 0$
$T_1 < t < 2T_1$	$V_+ = \frac{V_o}{2}$	$V_+ = \frac{V_o}{2}$	$V_+ = \frac{2}{3} V_o$
	$V_- = 0$	$V_- = \frac{V_o}{6}$	$V_- = 0$
$2T_1 < t < 3T_1$	$V_+ = \frac{V_o}{2}$	$V_+ = \frac{V_o}{2}$	$V_+ = \frac{2}{3} V_o$
	$V_- = \frac{V_o}{6}$	$V_- = \frac{V_o}{6}$	$V_- = 0$



b)

t	$z = 0$	$z = \ell_{1-}$	$z = \ell_{1+}$
$0 < t < T_1$	$V_+ = \frac{V_o}{2}$	$V_+ = 0$	$V_+ = 0$
	$V_- = 0$	$V_- = 0$	$V_- = 0$
$T_1 < t < 2T_1$	$V_+ = \frac{V_o}{2}$	$V_+ = \frac{V_o}{2}$	$V_+ = \frac{V_o}{2}$
	$V_- = 0$	$V_- = 0$	$V_- = 0$



Section 8.3

16. a) $\hat{v}(z) = V_+e^{-jkz} + V_-e^{jkz}$

$$\hat{i}(z) = Y_o[V_+e^{-jkz} - V_-e^{jkz}]$$

$$\hat{v}(z=0) = \hat{i}(z=0)jX \to V_+ + V_- = jXY_o[V_+ - V_-]$$

$$\hat{v}(z=-\ell) = V_o = V_+e^{jk\ell} + V_-e^{-jk\ell}$$

$$V_+ = \frac{-V_o(1 + jXY_o)}{[(1-jXY_o)e^{-jk\ell} - (1+jXY_o)e^{jk\ell}]}, \quad V_- = \frac{V_o(1 - jXY_o)}{[(1-jXY_o)e^{-jk\ell} - (1+jXY_o)e^{jk\ell}]}$$

b) Resonance

$$(1-jXY_o)e^{-jk\ell} = (1+jXY_o)e^{jk\ell} \to \tan k\ell = -XY_o; \quad k = \frac{\omega}{c}$$

$$|X| = Z_o \to \tan k\ell = \mp 1 \begin{array}{l}\text{inductive}\\ \text{capacitive}\end{array} \to k\ell = \begin{array}{l}\frac{3}{4}\pi + n\pi\\ \frac{\pi}{4} + n\pi, \quad n = 0, 1, 2, \ldots\end{array}$$

$$\omega_n = \begin{cases} \frac{c}{\ell}\,(n + \frac{3}{4})\pi & \text{inductive} \\ \frac{c}{\ell}\,(n + \frac{1}{4})\pi & \text{capacitive} \end{cases}$$

c) $\hat{i}(z=-\ell) = I_o = Y_o[V_+ e^{jk\ell} - V_- e^{-jk\ell}]$

$$V_+ = \frac{I_o Z_o(1 + jXY_o)}{[(1-jXY_o)e^{-jk\ell} + (1+jXY_o)e^{jk\ell}]};\quad V_- = \frac{-I_o Z_o(1-jXY_o)}{[(1-jXY_o)e^{-jk\ell} + (1+jXY_o)e^{jk\ell}]}$$

Resonance

$$(1-jXY_o)e^{-jk\ell} + (1+jXY_o)e^{jk\ell} = 0 \rightarrow \tan k\ell = \frac{1}{XY_o}$$

$$|X| = Z_o \rightarrow \tan k\ell = \pm 1 \quad \begin{matrix}\text{inductive}\\ \text{capacitive}\end{matrix} \rightarrow \omega_n = \begin{cases} \frac{c}{\ell}\,(n+\frac{1}{4})\pi \\ \frac{c}{\ell}\,(n+\frac{3}{4})\pi \end{cases} \qquad n = 0, 1, 2, \ldots$$

17. a) $R = \dfrac{1}{\sigma_w \delta 2\pi a} + \dfrac{1}{\sigma_w \delta 2\pi b},\quad G = \dfrac{2\pi\sigma}{\ell n \frac{b}{a}}$

b) $\alpha = \dfrac{1}{2}\,[RY_o + GZ_o];\quad Y_o = \dfrac{1}{Z_o} = \sqrt{\dfrac{\varepsilon}{\mu}}\,\dfrac{2\pi}{\ell n \frac{b}{a}}$

$$= \frac{1}{2}\left\{\frac{\sqrt{\varepsilon/\mu}\ (\frac{1}{a} + \frac{1}{b})}{\sigma_w \delta \ell n \frac{b}{a}} + \sigma\sqrt{\frac{\mu}{\varepsilon}}\right\}$$

c) $\sigma = 0 \rightarrow \alpha = \dfrac{\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,(\frac{1}{a} + \frac{1}{b})}{\sigma_w \delta \ell n \frac{b}{a}}$

$$\frac{d\alpha}{da} = \frac{\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}}{\sigma_w \delta}\left[\frac{-1}{a^2 \ell n \frac{b}{a}} + \frac{(\frac{1}{a} + \frac{1}{b})}{(\ell n \frac{b}{a})^2}\,\frac{a}{b}\,\frac{b}{a^2}\right] = 0$$

$$\frac{a(\frac{1}{a} + \frac{1}{b})}{\ell n \frac{b}{a}} = 1 \rightarrow \ell n \frac{b}{a} = 1 + \frac{a}{b} \rightarrow \frac{b}{a} \approx 3.6$$

18. a) $\hat{v}(z) = V_+ e^{-jkz} + V_- e^{jkz}$

$$\hat{i}(z) = Y_o[V_+ e^{-jkz} - V_- e^{jkz}]$$

$$\hat{v}(z=-\ell) = V_o = V_+ e^{jk\ell} + V_- e^{-jk\ell}$$

$$\hat{v}(z=0) = \hat{i}(z=0)R_L \rightarrow V_+ + V_- = Y_o R_L (V_+ - V_-)$$

$$V_+ = \frac{-V_o(1 + Y_o R_L)}{[e^{-jk\ell}(1 - Y_o R_L) - (1 + Y_o R_L)e^{jk\ell}]}; \quad V_- = \frac{V_o(1 - Y_o R_L)}{[e^{-jk\ell}(1 - Y_o R_L) - (1 + Y_o R_L)e^{jk\ell}]}$$

b) $$\lim_{k\ell \ll 1} \quad V_+ \approx \frac{V_o(1 + Y_o R_L)}{2Y_o R_L + 2jk\ell}, \quad V_- \approx \frac{-V_o(1 - Y_o R_L)}{2Y_o R_L + 2jk\ell}$$

$$\hat{v}(z) \approx \frac{V_o}{2(Y_o R_L + jk\ell)} [(1 + Y_o R_L)(1 - jkz) - (1 - Y_o R_L)(1 + jkz)]$$

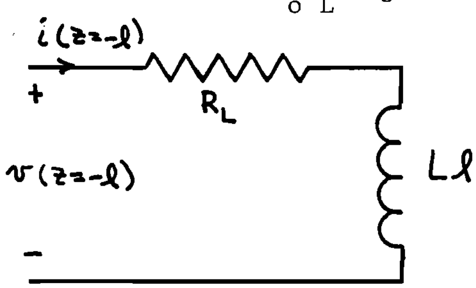
$$\approx \frac{V_o}{(Y_o R_L + jk\ell)} [Y_o R_L - jkz]$$

$$\hat{i}(z) \approx \frac{Y_o V_o}{2(Y_o R_L + jk\ell)} [(1 + Y_o R_L)(1 - jkz) + (1 - Y_o R_L)(1 + jkz)]$$

$$\approx \frac{Y_o V_o}{(Y_o R_L + jk\ell)} [1 - jkz Y_o R_L]$$

c) $Y_o R_L \ll k\ell$

$$\hat{i}(z=-\ell) = \frac{Y_o V_o(1 + jk\ell Y_o R_L)}{Y_o R_L + jk\ell} \rightarrow V_o = \frac{\hat{i}(z=-\ell)[Y_o R_L + jk\ell]Z_o}{1 + jk\ell Y_o R_L}$$



$$\approx \hat{i}(z=-\ell)Z_o[Y_o R_L + jk\ell][1 - jk\ell Y_o R_L]$$

$$\approx \hat{i}(z=-\ell)Z_o[Y_o R_L + jk\ell]$$

$$\approx \hat{i}(z=-\ell)[R_L + j\sqrt{L/C}\,\omega\sqrt{LC}\,\ell]$$

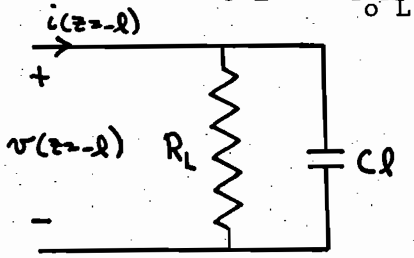
$$\approx \hat{i}(z=-\ell)[R_L + j\omega L\ell]$$

$Y_o R_L \gg k\ell$

$$\hat{i}(z=-\ell) \approx \frac{Y_o V_o(1 + jk\ell Y_o R_L)}{Y_o R_L(1 + \frac{jk\ell}{Y_o R_L})} \approx \frac{V_o}{R_L}(1 + jk\ell Y_o R_L)(1 - \frac{jk\ell}{Y_o R_L})$$

$$\approx V_o(\frac{1}{R_L} + jk\ell Y_o)$$

$$\approx V_o(\frac{1}{R_L} + j\omega C\ell)$$



Section 8.4

19. a) $Z_n(z=0) = 2(1-j)$, $Z_n(z=-\frac{\lambda}{4}) = \frac{1}{2(1-j)} \rightarrow Y_n(z=-\frac{\lambda}{4}) = 2(1-j)$

$$Y_T(z=-\frac{\lambda}{4}) = jB + \frac{2(1-j)}{50} \rightarrow B = .04,\ R_s = 25$$

b) $\langle P_L \rangle = \frac{1}{4}\frac{V_o^2}{R_s}$

20. a) Switch open $\rightarrow Z(z=-\frac{\lambda}{2}) = 400 \rightarrow \langle P \rangle = \frac{V_o^2}{1600}$

Switch closed $\rightarrow Z(z=-\frac{\lambda}{2}) = 800 \rightarrow \langle P \rangle = \frac{V_o^2}{2400}$

b) Switch open $\rightarrow \langle P_L \rangle = \frac{1}{2}\langle P \rangle = \frac{V_o^2}{3200}$

Switch closed $\rightarrow \langle P_L \rangle = \frac{V_o^2}{7200}$

c) Switch open $\rightarrow Z_o = 50$ line, VSWR $= \frac{R_L}{Z_o} = 2$

$Z_o = 100$ line, VSWR $= \frac{Z_o}{25} = 4$

Switch closed $\rightarrow Z_o = 50$ line, VSWR $= \frac{R_L}{Z_o} = 2$

$Z_o = 100$ line, VSWR $= \frac{Z_o}{12.5} = 8$

21. a) and b) $i(t) = \mathrm{Re}\,\hat{I}e^{j\omega t}$; $\hat{I} = |\hat{I}|e^{j\phi} \rightarrow i(t) = |\hat{I}|\cos(\omega t + \phi)$

$$\hat{I} = \frac{V_o}{Z_o Z_n(-\ell)}$$

ℓ	$Z_n(-\ell)$	$\hat{I}$	$\lvert\hat{I}\rvert$	ϕ	$\langle P\rangle = \frac{1}{2} V_o \lvert\hat{I}\rvert \cos\phi$
$\frac{\lambda}{8}$	$.2 - .4j$	$\frac{V_o}{50(.2-.4j)}$	$\frac{V_o}{50\sqrt{.2}}$	$63.4°$	$\frac{V_o^2}{100\sqrt{.2}}(.447)$
$\frac{\lambda}{4}$	$.2 + .4j$	$\frac{V_o}{50(.2+.4j)}$	$\frac{V_o}{50\sqrt{.2}}$	$-63.4°$	$\frac{V_o^2}{100\sqrt{.2}}(.447)$
$\frac{3\lambda}{8}$	$1 + 2j$	$\frac{V_o}{50(1+2j)}$	$\frac{V_o}{50\sqrt{5}}$	$-63.4°$	$\frac{V_o^2}{100\sqrt{5}}(.447)$
$\frac{\lambda}{2}$	$1 - 2j$	$\frac{V_o}{50(1-2j)}$	$\frac{V_o}{50\sqrt{5}}$	$63.4°$	$\frac{V_o^2}{100\sqrt{5}}(.447)$

c) $$\Gamma_L = \frac{Z_{nL} - 1}{Z_{nL} + 1} = \frac{-2j}{2(1-j)} = \frac{-j}{1-j}$$

$$|\Gamma_L| = \frac{1}{\sqrt{2}} \rightarrow \text{VSWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1+\sqrt{2}}{\sqrt{2}-1} \approx 5.83$$

22. a) $$\hat{v}(z) = V_+ e^{-jkz} + V_- e^{jkz}$$

$$\hat{i}(z) = Y_o[V_+ e^{-jkz} - V_- e^{jkz}]$$

$$\hat{v}(z=0) = \hat{i}(z=0)R_L \rightarrow V_+ + V_- = R_L Y_o(V_+ - V_-) \rightarrow \frac{V_-}{V_+} = \Gamma_L = \frac{R_L Y_o - 1}{R_L Y_o + 1} = \frac{1}{3}$$

$$\hat{v}(z=-\ell) = V_o = V_+ e^{jk\ell} + V_- e^{-jk\ell}; \quad k\ell = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$

$$= j(V_+ - V_-)$$

$$= \frac{2}{3} jV_+$$

$$V_+ = -\frac{3}{2} jV_o, \quad V_- = -\frac{1}{2} jV_o$$

$$\hat{v}(z) = -\frac{jV_o}{2}[3e^{-jkz} + e^{jkz}]$$

$$\hat{i}(z) = -j\frac{V_o Y_o}{2}[3e^{-jkz} - e^{jkz}]$$

b) $\text{VSWR} = \dfrac{R_L}{Z_o} = 2$

c) $$|\hat{v}(z)|^2 = \left(\frac{V_o}{2}\right)^2[3e^{-jkz} + e^{jkz}][3e^{jkz} + e^{-jkz}]$$

$$= \left(\frac{V_o}{2}\right)^2[10 + 3(e^{-2jkz} + e^{2jkz})]$$

$$= \left(\frac{V_o}{2}\right)^2[10 + 6\cos 2kz]$$

$$|\hat{v}(z)|_{max} \rightarrow kz = -n\pi,\ n = 0, 1, 2, \ldots$$

$$\text{Since } -\frac{\lambda}{4} < z < 0 \rightarrow n = 0 \rightarrow z = 0 \rightarrow |\hat{v}(z)|_{max} = 2V_o$$

$$|\hat{v}(z)|_{min} \rightarrow kz = -\frac{(2n+1)\pi}{2},\ n = 0, 1, 2, \ldots$$

$$n = 0 \rightarrow z = -\frac{\lambda}{4} \rightarrow |\hat{v}(z)|_{min} = V_o$$

$$\text{Check: } \text{VSWR} = \frac{|\hat{v}(z)|_{max}}{|\hat{v}(z)|_{min}} = 2$$

23. $\text{VSWR} = 3,\ Z_o = 100,\ \dfrac{\lambda}{2} = .5 \text{ meters},\ d_{min} = .2 \text{ meters}$

$$kd_{min} = \frac{2\pi d_{min}}{\lambda} = .4\pi$$

$$|\Gamma_L| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} = \frac{1}{2},\ \phi = \pi\left[\frac{4d_{min}}{\lambda} - 1\right] = -.2\pi$$

$$\Gamma_L = |\Gamma_L|e^{j\phi} = \frac{1}{2}e^{-j(.2\pi)}$$

$$Z_L = Z_o\frac{[\text{VSWR} - j\tan\frac{\phi}{2}]}{[1 - j\text{VSWR}\tan\frac{\phi}{2}]};\ \tan\frac{\phi}{2} = -.3249$$

$$= \frac{100[3 + .3249j]}{[1 + j(3)(.3249)]}$$

$$= 170.08 - 133.29j$$

24. a) $A \rightarrow Z_{nL} = 2(1-j)$, $B \rightarrow Y_{nL} = .25 + .25j$

$C \rightarrow Y_n(-\ell_1) = 1 + 1.6j$, $\ell_1 = \ell_D = .137\lambda$

$Y_{n2} = -1.6j \rightarrow \ell_2 = .089\lambda$

$D \rightarrow Y_n(-\ell_1) = 1 - 1.6j$, $\ell_1 = \ell_D + \ell_C = .279\lambda$

$Y_{n2} = +1.6j \rightarrow \ell_2 = .411\lambda$

$\ell_1 = .137\lambda + \frac{n\lambda}{2}$, $\ell_2 = .089\lambda + \frac{m\lambda}{2}$

$\ell_1 = .279\lambda + \frac{n\lambda}{2}$, $\ell_2 = .411\lambda + \frac{m\lambda}{2}$

b) $\ell_1 = .25\lambda + \frac{n\lambda}{2}$, $\ell_2 = .074\lambda + \frac{m\lambda}{2}$

$\ell_1 = .3755\lambda + \frac{n\lambda}{2}$, $\ell_2 = .426\lambda + \frac{m\lambda}{2}$

c) $\ell_1 = .039\lambda + \frac{n\lambda}{2}$, $\ell_2 = .177\lambda + \frac{m\lambda}{2}$

$\ell_1 = .25\lambda + \frac{n\lambda}{2}$, $\ell_2 = .324\lambda + \frac{m\lambda}{2}$

d) $\ell_1 = .294\lambda + \frac{n\lambda}{2}$, $\ell_2 = .089\lambda + \frac{m\lambda}{2}$

$\ell_1 = .446\lambda + \frac{n\lambda}{2}$, $\ell_2 = .411\lambda + \frac{m\lambda}{2}$

25. a) $Z_{nL} = 2(1 - j) \rightarrow Y_{nL} = .25 + .25j$

$Y_{a1} = .25 - j.3375$

$Y_{a2} = .25 - j1.65$

$$Y_1 = j\mathrm{Im}(Y_a - Y_{n\ell}) = \begin{cases} -j.5875 \rightarrow \ell_1 = .166\lambda \\ -j1.90 \rightarrow \ell_1 = .077\lambda \end{cases}$$

$Y_{b1} = 1 - j1.6$

$Y_{b2} = 1 + j3.6$

$$Y_2 = -j\mathrm{Im}Y_b = \begin{cases} j1.6 \rightarrow \ell_2 = .411\lambda \\ -j3.6 \rightarrow \ell_2 = .043\lambda \end{cases}$$

$\ell_1 = .166\lambda + \frac{n\lambda}{2},\ \ell_2 = .411\lambda + \frac{m\lambda}{2}$

$\ell_1 = .077\lambda + \frac{n\lambda}{2},\ \ell_2 = .043\lambda + \frac{m\lambda}{2}$

b) $Z_{nL} = 1 + 2j \rightarrow Y_{nL} = .2 - j.4 = Y_{a1}$

$Y_{a2} = .2 - j1.6$

$$Y_1 = j\mathrm{Im}(Y_a - Y_{nL}) = \begin{cases} j0 \rightarrow \ell_1 = .25\lambda \\ -j1.2 \rightarrow \ell_1 = .11\lambda \end{cases}$$

$Y_{b1} = 1 - j2.0,\ Y_{b2} = 1 + j3.8$

$$Y_2 = -j\mathrm{Im}Y_b = \begin{cases} j2.0 \rightarrow \ell_2 = .426\lambda \\ -j3.8 \rightarrow \ell_2 = .041\lambda \end{cases}$$

$\ell_1 = .25\lambda + \frac{n\lambda}{2},\ \ell_2 = .426\lambda + \frac{m\lambda}{2}$

$\ell_1 = .11\lambda + \frac{n\lambda}{2},\ \ell_2 = .041\lambda + \frac{m\lambda}{2}$

c) $Z_{nL} = 1 - j.5 \rightarrow Y_{nL} = .8 + j.4$

$Y_{a1} = .8 - j.025$

$Y_{a2} = .8 - j2.0$

$$Y_1 = j\mathrm{Im}(Y_a - Y_{nL}) = \begin{cases} -j.425 \rightarrow \ell_1 = .187\lambda \\ -j2.4 \rightarrow \ell_1 = .063\lambda \end{cases}$$

$Y_{b1} = 1 - j.25,\ Y_{b2} = 1 + j2.2$

$$Y_2 = -j\mathrm{Im}Y_b = \begin{cases} j.25 \rightarrow \ell_2 = .289\lambda \\ -j2.2 \rightarrow \ell_2 = .068\lambda \end{cases}$$

$\ell_1 = .187\lambda + \frac{n\lambda}{2},\ \ell_2 = .289\lambda + \frac{m\lambda}{2}$

$\ell_1 = .063\lambda + \frac{n\lambda}{2},\ \ell_2 = .068\lambda + \frac{m\lambda}{2}$

d) $Z_{nL} = .5 + j1 \rightarrow Y_{nL} = .4 - j.8$

$Y_{a1} = .4 - j.2,\ Y_{a2} = .4 - j1.8$

$$Y_1 = j\mathrm{Im}(Y_a - Y_{nL}) = \begin{cases} +j.6 \rightarrow \ell_1 = .334\lambda \\ -j1.0 \rightarrow \ell_1 = .125\lambda \end{cases}$$

$Y_{b1} = 1 - j,\ Y_{b2} = 1 + j3.0$

$$Y_2 = -j\mathrm{Im}Y_b = \begin{cases} j \rightarrow \ell_2 = .375\lambda \\ -j3.0 \rightarrow \ell_2 = .051\lambda \end{cases}$$

$$\ell_1 = .334\lambda + \frac{n\lambda}{2},\quad \ell_2 = .375\lambda + \frac{m\lambda}{2}$$

$$\ell_1 = .125\lambda + \frac{n\lambda}{2},\quad \ell_2 = .051\lambda + \frac{m\lambda}{2}$$

26. a) Inductive

$Z_n(z=0) = 1 + j$

$$Z_n\left(z=-\frac{\lambda}{4}\right) = \frac{1}{1+j} = \frac{(1-j)}{2}$$

$$Z\left(z=-\frac{\lambda}{4}\right) = \frac{Z_o}{2}(1-j)$$

Capacitive

$Z_n(z=0) = 1 - j$

$$Z_n\left(z=-\frac{\lambda}{4}\right) = \frac{1}{1-j} = \frac{1+j}{2}$$

$$Z\left(z=-\frac{\lambda}{4}\right) = \frac{Z_o}{2}(1+j)$$

b) Inductive

$$\hat{i}\left(z=-\frac{\lambda}{4}\right) = \frac{2V_o}{Z_o(1-j)} = \frac{V_o(1+j)}{Z_o}$$

$$i\left(z=-\frac{\lambda}{4},t\right) = \mathrm{Re}\,\hat{i}\left(z=-\frac{\lambda}{4}\right)e^{j\omega t}$$

$$= \frac{\sqrt{2}\,V_o}{Z_o}\cos(\omega t+45°)$$

Capacitive

$$\hat{i}\left(z=-\frac{\lambda}{4}\right) = \frac{2V_o}{Z_o(1+j)} = \frac{V_o(1-j)}{Z_o}$$

$$i\left(z=-\frac{\lambda}{4},t\right) = \mathrm{Re}\,\hat{i}\left(z=-\frac{\lambda}{4}\right)e^{j\omega t}$$

$$= \frac{\sqrt{2}\,V_o}{Z_o}\cos(\omega t-45°)$$

c) $\omega_o \rightarrow 2\omega_o$

Inductive

$Z_n(z=0) = 1 + 2j$

$$Z_n\left(z=-\frac{\lambda}{4}\right) = \frac{1}{1+2j} = \frac{(1-2j)}{5}$$

$$Z\left(z=-\frac{\lambda}{4}\right) = \frac{Z_o(1-2j)}{5}$$

Capacitive

$Z_n(z=0) = 1 - \frac{1}{2}j$

$$Z_n\left(z=-\frac{\lambda}{4}\right) = \frac{1}{1-\frac{1}{2}j} = \frac{1+\frac{1}{2}j}{1.25}$$

$$Z\left(z=-\frac{\lambda}{4}\right) = \frac{Z_o(1+\frac{1}{2}j)}{1.25}$$

Inductive

$$\hat{i}(z=-\tfrac{\lambda}{4}) = \frac{5V_o}{Z_o(1-2j)} = \frac{V_o(1+2j)}{Z_o}$$

$$i(z=-\tfrac{\lambda}{4},t) = \frac{V_o\sqrt{5}}{Z_o}\cos(\omega t+63.43°)$$

Capacitive

$$\hat{i}(z=-\tfrac{\lambda}{4}) = \frac{(1-\frac{1}{2}j)V_o}{Z_o}$$

$$i(z=-\tfrac{\lambda}{4},t) = \frac{V_o\sqrt{1.25}}{Z_o}\cos(\omega t-26.57°)$$

d) $\omega_o \to \frac{1}{2}\omega_o$

Inductive

$$Z_n(z=0) = 1 + \frac{j}{2}$$

$$Z_n(z=-\tfrac{\lambda}{4}) = \frac{1}{1+\frac{j}{2}} = \frac{(1-\frac{1}{2}j)}{1.25}$$

$$Z(z=-\tfrac{\lambda}{4}) = \frac{Z_o(1-\frac{1}{2}j)}{1.25}$$

$$\hat{i}(z=-\tfrac{\lambda}{4})=\frac{V_o}{Z_o}(1+\tfrac{j}{2})$$

$$i(z=-\tfrac{\lambda}{4},t) = \frac{V_o\sqrt{1.25}}{Z_o}\cos(\omega t+26.57°)$$

Capacitive

$$Z_n(z=0) = 1 - 2j$$

$$Z_n(z=-\tfrac{\lambda}{4}) = \frac{1}{1-2j} = \frac{(1+2j)}{5}$$

$$Z(z=-\tfrac{\lambda}{4}) = \frac{Z_o(1+2j)}{5}$$

$$\hat{i}(z=-\tfrac{\lambda}{4}) = \frac{V_o(1-2j)}{Z_o}$$

$$i(z=-\tfrac{\lambda}{4},t) = \frac{V_o\sqrt{5}}{Z_o}\cos(\omega t-63.43°)$$

Section 8.6

27. a) $\nabla\times\overline{E} = -\mu\dfrac{\partial\overline{H}}{\partial t}$

$$\nabla\times\overline{H} = \overline{J}_f + \varepsilon\frac{\partial\overline{E}}{\partial t};\quad \frac{\partial\overline{J}_f}{\partial t} = \omega_p^2\varepsilon\overline{E}$$

$$\varepsilon\nabla\cdot\overline{E} = 0$$

$$\mu\nabla\cdot\overline{H} = 0$$

$$\nabla\times(\nabla\times\overline{E}) = \cancel{\nabla(\nabla\cdot\overline{E})}^{0} - \nabla^2\overline{E} = -\mu\frac{\partial}{\partial t}(\nabla\times\overline{H}) = -\mu\left[\omega_p^2\varepsilon\overline{E} + \varepsilon\frac{\partial^2\overline{E}}{\partial t^2}\right]$$

$$\nabla^2\overline{E} = \frac{1}{c^2}\frac{\partial^2\overline{E}}{\partial t^2} + \frac{\omega_p^2}{c^2}\overline{E}$$

$$\nabla\times(\nabla\times\overline{H}) = \cancel{\nabla(\nabla\cdot\overline{H})}^{0} - \nabla^2\overline{H} = \nabla\times\overline{J}_f + \varepsilon\frac{\partial}{\partial t}\nabla\times\overline{E}$$

$$= \nabla\times\overline{J}_f - \varepsilon\mu\frac{\partial^2\overline{H}}{\partial t^2}$$

$$\nabla^2 \frac{\partial \overline{H}}{\partial t} = -\nabla \times \frac{\partial \overline{J}_f}{\partial t} + \varepsilon\mu \frac{\partial^3 \overline{H}}{\partial t^3} = -\omega_p^2 \varepsilon \nabla \times \overline{E} + \varepsilon\mu \frac{\partial^3 \overline{H}}{\partial t^3}$$

$$= \frac{\omega_p^2}{c^2} \frac{\partial \overline{H}}{\partial t} + \frac{1}{c^2} \frac{\partial^3 \overline{H}}{\partial t^3}$$

$$\nabla^2 \overline{H} = \frac{1}{c^2} \frac{\partial^2 \overline{H}}{\partial t^2} + \frac{\omega_p^2}{c^2} \overline{H}$$

Solutions the same as in text with $k_x^2 + k_y^2 + k_z^2 = \dfrac{\omega^2 - \omega_p^2}{c^2}$

TM - Eqs. (13), (18), (19), (22)

TE - Eqs. (30), (32)

b) $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b} \rightarrow k_z = \left[\frac{\omega^2 - \omega_p^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$

$$\omega_{co} = \left[\omega_p^2 + \left(\frac{\pi c}{a}\right)^2\right]^{1/2}$$

c) $v_p = \frac{\omega}{k_z}$, $v_g = \frac{d\omega}{dk_z} = \frac{k_z c^2}{\omega}$

d) TM [From (42) and (43)]

$$\langle P_z \rangle = \frac{\omega\varepsilon k_z ab E_o^2}{8(k_x^2 + k_y^2)} \qquad \omega > \omega_{co}$$

TE [From (47)]

$$\langle P_z \rangle = \begin{cases} \dfrac{\omega\mu k_z ab H_o^2}{8(k_x^2 + k_y^2)} & m, n \neq 0 \\[2ex] \dfrac{\omega\mu k_z ab H_o^2}{4(k_x^2 + k_y^2)} & m \text{ or } n = 0 \end{cases} \qquad \omega > \omega_{co}$$

e) From (54)

$$\alpha = \frac{2\left(\frac{\pi}{a}\right)^2\left[b + \frac{a}{2}\left(\frac{\omega^2 a^2}{\pi^2 c^2}\right)\right]}{\omega\mu ab k_z \sigma_w \delta} \; ; \qquad k_z = \left[\frac{\omega^2 - \omega_p^2}{c^2} - \left(\frac{\pi}{a}\right)^2\right]^{1/2}$$

28. a)

$$\langle P_z \rangle = \frac{\omega\varepsilon k_z abE_o^2}{8(k_x^2 + k_y^2)}; \quad k_x = \frac{m\pi}{a}, \ k_y = \frac{n\pi}{b}$$

$$|\hat{K}_z(x,y=0)| = |\hat{K}_z(x,y=b)| = \frac{k_y k^2 E_o \sin k_x x}{j\omega\mu(k_x^2 + k_y^2)}$$

$$|\hat{K}_z(x=0,y)| = |\hat{K}_z(x=a,y)| = \frac{k_x k^2 E_o \sin k_y y}{j\omega\mu(k_x^2 + k_y^2)}$$

$$\langle P_d(x=0,y)\rangle = \langle P_d(x=a,y)\rangle = \frac{1}{2}\,\mathrm{Re}(\hat{\bar{E}}\cdot\hat{\bar{K}}^*) = \frac{1}{2}\,\frac{|\hat{K}_z(x=0,y)|^2}{\sigma_w\delta}$$

$$\langle P_d(x,y=0)\rangle = \langle P_d(x,y=b)\rangle = \frac{1}{2}\,\mathrm{Re}(\hat{\bar{E}}\cdot\hat{\bar{K}}^*) = \frac{1}{2}\,\frac{|\hat{K}_z(x,y=0)|^2}{\sigma_w\delta}$$

$$\langle P_{dL}\rangle = \int_0^b 2\langle P_d(x=0,y)\rangle dy + \int_0^a 2\langle P_d(x,y=0)\rangle dx$$

$$= \frac{k^4E_o^2}{\omega^2\mu^2(k_x^2 + k_y^2)^2\sigma_w\delta}\left\{\int_0^b k_x^2\sin^2 k_y y dy + \int_0^a k_y^2 \sin^2 k_x x dx\right\}$$

$$= \frac{\omega^2\varepsilon^2E_o^2}{\sigma_w\delta(k_x^2 + k_y^2)^2}\left\{\frac{k_x^2 b}{2} + \frac{k_y^2 a}{2}\right\}$$

b) $$\alpha = \frac{1}{2}\,\frac{\langle P_{dL}\rangle}{\langle P\rangle} = \frac{2\omega\varepsilon(bk_x^2 + ak_y^2)}{\sigma_w\delta k_z ab(k_x^2 + k_y^2)}$$

29. a) <u>TE</u>

<u>Electric Field Lines</u>

$$\frac{dy}{dx} = \frac{E_y}{E_x} = -\frac{k_x}{k_y}\,\frac{\tan k_x x}{\tan k_y y} \rightarrow \tan k_y y \; d(k_y y) = -\tan k_x x \; d(k_x x)$$

$$-\ln\cos k_y y = \ln\cos k_x x + \text{constant} \rightarrow \cos k_x x \, \cos k_y y = \text{constant}$$

Magnetic Field Lines

$$\frac{dy}{dx} = \frac{H_y}{H_x} = \frac{k_y}{k_x}\frac{\tan k_y y}{\tan k_x x} \rightarrow \frac{\cot k_y y\, d(k_y y)}{k_y^2} = \frac{\cot k_x x\, d(k_x x)}{k_x^2}$$

$$\frac{1}{k_y^2}\ln \sin k_y y = \frac{1}{k_x^2}\ln \sin k_x x + \text{constant}$$

$$\frac{(\sin k_x x)^{(k_y/k_x)^2}}{\sin k_y y} = \text{constant}$$

TM

Electric Field Lines

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{k_y}{k_x}\frac{\tan k_x x}{\tan k_y y} \rightarrow \frac{\tan k_y y\, d(k_y y)}{k_y^2} = \frac{\tan k_x x\, d(k_x x)}{k_x^2}$$

$$\frac{1}{k_y^2}\ln \cos k_y y = \frac{1}{k_x^2}\ln \cos k_x x + \text{constant}$$

$$\frac{(\cos k_x x)^{(k_y/k_x)^2}}{\cos k_y y} = \text{constant}$$

Magnetic Field Lines

$$\frac{dy}{dx} = \frac{H_y}{H_x} = -\frac{k_x}{k_y}\frac{\cot k_x x}{\cot k_y y} \rightarrow \cot k_y y\, d(k_y y) = -\cot k_x x\, d(k_x x)$$

$$\ln \sin k_y y = -\ln \sin k_x x + \text{constant}$$

$$\sin k_x x \sin k_y y = \text{constant}$$

b) $$\hat{\overline{K}}(x, y=0) = H_o\left[\overline{i}_x \cos k_x x - \overline{i}_z \frac{j k_z k_x \sin k_x x}{k_x^2 + k_y^2}\right]$$

$$\hat{\overline{K}}(x, y=b) = -H_o \cos n\pi\left[\overline{i}_x \cos k_x x - \overline{i}_z \frac{j k_z k_x}{k_x^2 + k_y^2} \sin k_x x\right]$$

$$\hat{\overline{K}}(x=0, y) = H_o\left[-\overline{i}_y \cos k_y y + \overline{i}_z \frac{j k_z k_y}{k_x^2 + k_y^2} \sin k_y y\right]$$

$$\hat{\overline{K}}(x=a,y) = -H_o \cos m\pi[-\overline{i}_y \cos k_y y + \overline{i}_z \frac{jk_y k_z}{k_x^2 + k_y^2} \sin k_y y]$$

On y = 0 and y = b surfaces

$$\frac{dz}{dx} = \frac{K_z}{K_x} = \frac{-k_z k_x}{k_x^2 + k_y^2} \tan k_x x \tan k_z z$$

$$\cot k_z z \, d(k_z z) = \frac{-k_z^2}{k_x^2 + k_y^2} \tan k_x x \, d(k_x x)$$

$$\ell n \sin k_z z = \frac{+k_z^2}{k_x^2 + k_y^2} \ell n \cos k_x x + \text{constant}$$

$$\frac{(\cos k_x x)^{k_z^2/(k_x^2 + k_y^2)}}{\sin k_z z} = \text{constant}$$

On x = 0 and x = a surfaces

$$\frac{dz}{dy} = \frac{-k_z k_y}{k_x^2 + k_y^2} \tan k_y y \tan k_z z$$

$$\frac{(\cos k_y y)^{k_z^2/(k_x^2 + k_y^2)}}{\sin k_z z} = \text{constant}$$

c) TM - At x = 0

$$\hat{\sigma}_f(x=0,y) = \frac{-jk_z k_x}{k_x^2 + k_y^2} \varepsilon E_o \sin k_y y$$

$$\hat{K}_z(x=0,y) = \frac{k_x k_z^2 E_o}{j\omega\mu(k_x^2 + k_y^2)} \sin k_y y$$

$$\nabla_\Sigma \cdot \overline{K} = \frac{\partial K_z}{\partial z} \rightarrow -jk_z \hat{K}_z$$

$$\frac{\partial \sigma_f}{\partial t} \rightarrow j\omega\hat{\sigma}_f$$

$$\nabla_\Sigma \cdot \overline{K} + \frac{\partial \sigma_f}{\partial t} = \frac{-k_z k_x}{(k_x^2 + k_y^2)} E_o \sin k_y y \left[\frac{k^2}{\omega\mu} - \omega\varepsilon\right] = 0$$

TE - At x = 0

$$\hat{\sigma}_f(x=0,y) = \frac{-k_y k^2 H_o}{j\omega(k_x^2 + k_y^2)} \sin k_y y$$

$$\hat{\overline{K}}(x=0,y) = H_o[\overline{i}_z \frac{jk_z k_y}{k_x^2 + k_y^2} \sin k_y y - \overline{i}_y \cos k_y y]$$

$$\nabla_\Sigma \cdot \overline{K} \rightarrow -jk_z \hat{K}_z + \frac{\partial}{\partial y} \hat{K}_y = H_o k_y \sin k_y y \left[\frac{k_z^2}{k_x^2 + k_y^2} + 1\right]$$

$$\nabla_\Sigma \cdot \overline{K} + \frac{\partial \sigma_f}{\partial t} = H_o k_y \sin k_y y \left[\frac{k_z^2}{k_x^2 + k_y^2} + 1 - \frac{k^2}{k_x^2 + k_y^2}\right] = 0$$

30. a) $f_c = \frac{\omega_c}{2\pi} = \frac{c}{2} [(\frac{m}{a})^2 + (\frac{n}{b})^2]^{1/2}$

$a = b \rightarrow f_c = \frac{c}{2a} [m^2 + n^2]^{1/2}$

$c = 3 \times 10^8$, $a = 10^{-2} \rightarrow f_c = 1.5 \times 10^{10} [m^2 + n^2]^{1/2}$

m	n	f_c
1	0	1.50×10^{10}
1	1	2.12×10^{10}
2	0	3×10^{10}
2	1	3.35×10^{10}
2	2	4.24×10^{10}
3	0	4.5×10^{10}
3	1	4.74×10^{10}
3	2	5.41×10^{10}
4	0	6.0×10^{10}
3	3	6.36×10^{10}

b) $f_{co} = \frac{c}{2a} \rightarrow a = \frac{c}{2f_{co}} = \frac{1.5 \times 10^8}{f_{co}}$

f_{co}	a
10^{10}	1.5 cm
10^{8}	1.5 m
10^{6}	150 m
10^{4}	15 km
10^{2}	1.5×10^{6} m

31. a) <u>TM</u>

$$\hat{E}_z(x,y,z) = E_o \sin k_x x \sin k_y y \sin k_z z; \; k_x = m\pi/a$$

$$\hat{E}_y(x,y,z) = \frac{E_o k_y k_z}{k_x^2 + k_y^2} \sin k_x x \cos k_y y \cos k_z z; \; k_y = n\pi/b$$

$$\hat{E}_x(x,y,z) = \frac{E_o k_x k_z}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \cos k_z z; \; k_z = p\pi/\ell$$

$$\hat{H}_x = -\frac{1}{j\omega\mu}\left(\frac{\partial \hat{E}_z}{\partial y} - \frac{\partial \hat{E}_y}{\partial z}\right)$$

$$= -\frac{E_o}{j\omega\mu} \sin k_x x \cos k_y y \sin k_z z\left(k_y + \frac{k_y k_z^2}{k_x^2 + k_y^2}\right)$$

$$= -\frac{E_o k_y}{j\omega\mu} \frac{k^2}{(k_x^2 + k_y^2)} \sin k_x x \cos k_y y \sin k_z z$$

$$\hat{H}_y = -\frac{1}{j\omega\mu}\left(\frac{\partial \hat{E}_x}{\partial z} - \frac{\partial \hat{E}_z}{\partial x}\right)$$

$$= \frac{E_o}{j\omega\mu} \cos k_x x \sin k_y y \sin k_z z \left(\frac{k_x k_z^2}{k_x^2 + k_y^2} + k_x\right)$$

$$= \frac{E_o k_x}{j\omega\mu} \frac{k^2}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \sin k_z z$$

$$\hat{H}_z = -\frac{1}{j\omega\mu}\left(\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y}\right) = 0$$

<u>TE</u>

$$\hat{H}_z(x,y,z) = H_o \cos k_x x \cos k_y y \sin k_z z$$

$$\hat{H}_y(x,y,z) = \frac{-H_o k_y k_z}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \cos k_z z$$

$$\hat{H}_x(x,y,z) = \frac{-H_o k_x k_z}{k_x^2 + k_y^2} \sin k_x x \cos k_y y \cos k_z z$$

$$\hat{E}_x = \frac{1}{j\omega\varepsilon}\left(\frac{\partial \hat{H}_z}{\partial y} - \frac{\partial \hat{H}_y}{\partial z}\right)$$

$$= \frac{-H_o}{j\omega\varepsilon} \cos k_x x \sin k_y y \sin k_z z\left(k_y + \frac{k_y k_z^2}{k_x^2 + k_y^2}\right)$$

$$= \frac{-H_o k^2 k_y}{j\omega\varepsilon(k_x^2 + k_y^2)} \cos k_x x \sin k_y y \sin k_z z$$

$$\hat{E}_y = \frac{1}{j\omega\varepsilon}\left(\frac{\partial \hat{H}_x}{\partial z} - \frac{\partial \hat{H}_z}{\partial x}\right)$$

$$= \frac{H_o}{j\omega\varepsilon} \sin k_x x \cos k_y y \sin k_z z\left(\frac{k_x k_z^2}{k_x^2 + k_y^2} + k_x\right)$$

$$= \frac{H_o k^2 k_x}{j\omega\varepsilon(k_x^2 + k_y^2)} \sin k_x x \cos k_y y \sin k_z z$$

$$\hat{E}_z = \frac{1}{j\omega\varepsilon}\left(\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y}\right) = 0$$

b) $k^2 = k_x^2 + k_y^2 + k_z^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{\ell}\right)^2 = \frac{\omega^2}{c^2}$

c) TE_{101}

$$\hat{\underline{E}} = \frac{j\omega\mu H_o}{k_x} \sin k_x x \sin k_z z\, \bar{i}_y; \quad k_x = \frac{\pi}{a},\ k_y = 0,\ k_z = \frac{\pi}{\ell}$$

$$\hat{\underline{H}} = H_o\left[-\frac{k_z}{k_x} \sin k_x x \cos k_z z\, \bar{i}_x + \cos k_x x \sin k_z z\, \bar{i}_z\right]$$

$$\hat{\underline{K}}(x,y=0,z) = -\hat{\underline{K}}(x,y=b,z) = -\hat{H}_x(x,y=0,z)\bar{i}_z + \hat{H}_z(x,y=0,z)\bar{i}_x$$

$$= H_o\left[\frac{k_z}{k_x} \sin k_x x \cos k_z z\, \bar{i}_z + \cos k_x x \sin k_z z\, \bar{i}_x\right]$$

$$\hat{\bar{K}}(x=0,y,z) = \hat{\bar{K}}(x=a,y,z) = -\hat{H}_z(x=0,y,z)\bar{i}_y$$

$$= -H_o \sin k_z z$$

$$<P_{dA}(x,y=0,z)> = <P_{dA}(x,y=b,z)> = \frac{1}{2}\,\mathrm{Re}(\hat{\bar{E}}\cdot\hat{\bar{K}}^*)$$

$$= \frac{1}{2}\frac{H_o^2}{\sigma_w\delta}[(\frac{k_z}{k_x})^2 \sin^2 k_x x \cos^2 k_z z + \cos^2 k_x x \sin^2 k_z z]$$

$$<P_{dA}(x=0,y,z)> = <P_{dA}(x=a,y\quad,z)> = \frac{1}{2}\frac{H_o^2}{\sigma_w\delta}\sin^2 k_z z$$

$$<P_d> = \int_{x=0}^{a}\int_{z=0}^{\ell} 2<P_{dA}(x,y=0,z)>dxdz + \int_{y=0}^{b}\int_{z=0}^{\ell} 2<P_{dA}(x=0,y,z)>dydz$$

$$= \frac{H_o^2}{\sigma_w\delta}\left\{(\frac{a}{\ell})^2\frac{a}{2}\frac{\ell}{2} + \frac{a}{2}\frac{\ell}{2} + \frac{b\ell}{2}\right\}$$

$$= \frac{H_o^2\ell}{2\sigma_w\delta}\left\{\frac{a}{2}(1 + (\frac{a}{\ell})^2) + b\right\}$$

d) $$<w> = \frac{1}{2}\varepsilon|\bar{E}|^2 + \frac{1}{2}\mu|\bar{H}|^2$$

$$= \frac{\omega^2\mu^2 H_o^2}{k_x^2}\sin^2 k_x x \sin^2 k_z z + \frac{1}{2}\mu H_o^2[(\frac{k_z}{k_x})^2 \sin^2 k_x x \cos^2 k_z z + \cos^2 k_x x \sin^2 k_z z]$$

$$= \frac{1}{2}\mu H_o^2\left\{\frac{\omega^2\varepsilon\mu}{k_x^2}\sin^2 k_x x \sin^2 k_z z + (\frac{a}{\ell})^2 \sin^2 k_x x \cos^2 k_z z + \cos^2 k_x x \sin^2 k_z z\right\}$$

$$<W> = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{\ell} <w>dxdydz$$

$$= \frac{1}{2}\mu H_o^2\left\{\frac{\omega^2\varepsilon\mu a^2}{\pi^2}\frac{a}{2}\frac{\ell b}{2} + (\frac{a}{\ell})^2\frac{a}{2}\frac{\ell b}{2} + \frac{a}{2}\frac{\ell b}{2}\right\}$$

$$= \frac{1}{8}\mu H_o^2 a\ell b\left\{\frac{\omega^2\varepsilon\mu a^2}{\pi^2} + (\frac{a}{\ell})^2 + 1\right\}$$

e) $$Q = \frac{\omega_o <W>}{<P_d>} = \frac{\omega_o ab[\frac{\omega_o^2\varepsilon\mu a^2}{\pi^2} + (\frac{a}{\ell})^2 + 1]\mu\sigma_w\delta}{[\frac{a}{2}(1 + (\frac{a}{\ell})^2) + b]}$$

$$= \frac{2ab}{\delta} \frac{\left[\frac{\omega_o^2 \varepsilon\mu a^2}{\pi^2} + \left(\frac{a}{\ell}\right)^2 + 1\right]}{\left[\frac{a}{2}\left(1 + \left(\frac{a}{\ell}\right)^2\right) + b\right]}$$

$$\delta = \sqrt{\frac{2}{\omega_o \mu\sigma_w}}, \quad \omega_o^2 = \pi^2 c^2 \left[\frac{1}{a^2} + \frac{1}{\ell^2}\right]$$

$$Q = \frac{4ab}{\delta} \frac{1}{\left[\frac{a}{2} + \frac{b\pi^2 c^2}{\omega_o^2 a^2}\right]}$$

<u>Section 8.7</u>

32. a) $k_x^2 + k_z^2 = \omega^2 \varepsilon\mu$

$$-\alpha^2 + k_z^2 = \omega^2 \varepsilon_o \mu_o$$

$$\alpha = 0 \rightarrow \omega^2 = \frac{k_x^2}{\varepsilon\mu - \varepsilon_o\mu_o}$$

<u>TM</u>

Odd: $\alpha = \frac{\varepsilon_o}{\varepsilon} k_x \tan k_x d = 0 \rightarrow k_x = \frac{n\pi}{d}$

Even: $\alpha = -\frac{\varepsilon_o}{\varepsilon} k_x \cot k_x d = 0 \rightarrow k_x = \frac{(2n+1)\pi}{2d}$

<u>TE</u>

Odd: $\alpha = \frac{\mu_o}{\mu} k_x \tan k_x d = 0 \rightarrow k_x = \frac{n\pi}{d}$

Even: $\alpha = -\frac{\mu_o}{\mu} k_x \cot k_x d = 0 \rightarrow k_x = \frac{(2n+1)\pi}{2d}$

b) $k_x d << 1 \rightarrow \tan k_x d \approx k_x d, \ \cot k_x d \approx \frac{1}{k_x d}$

<u>TM</u>

Odd: $\alpha \approx \frac{\varepsilon_o}{\varepsilon} k_x^2 d, \ k_x^2 + \alpha^2 = \omega^2(\varepsilon\mu - \varepsilon_o\mu_o)$

$$k_x^4 + k_x^2\left(\frac{\varepsilon}{\varepsilon_o d}\right)^2 - \omega^2(\varepsilon\mu - \varepsilon_o\mu_o)\left(\frac{\varepsilon}{\varepsilon_o d}\right)^2 = 0$$

$$k_x^2 = -\frac{1}{2}\left(\frac{\varepsilon}{\varepsilon_o d}\right)^2 \pm \sqrt{\frac{1}{4}\left(\frac{\varepsilon}{\varepsilon_o d}\right)^4 + \omega^2\left(\frac{\varepsilon}{\varepsilon_o d}\right)^2(\varepsilon\mu - \varepsilon_o\mu_o)}$$

$$k_z^2 = \omega^2\varepsilon_o\mu_o + \alpha^2 = \omega^2\varepsilon\mu - k_x^2$$

Even: $\alpha \approx -\frac{\varepsilon_o}{\varepsilon d}$, $k_x^2 = -\alpha^2 + \omega^2(\varepsilon\mu - \varepsilon_o\mu_o) = \omega^2(\varepsilon\mu - \varepsilon_o\mu_o) - \left(\frac{\varepsilon_o}{\varepsilon d}\right)^2$

<u>TE</u>

Odd: $\alpha \approx \frac{\mu_o}{\mu} k_x^2 d$, $k_x^2 = -\frac{1}{2}\left(\frac{\mu}{\mu_o d}\right)^2 \pm \sqrt{\frac{1}{4}\left(\frac{\mu}{\mu_o d}\right)^4 + \omega^2\left(\frac{\mu}{\mu_o d}\right)^2(\varepsilon\mu - \varepsilon_o\mu_o)}$

$$k_z^2 = \omega^2\varepsilon_o\mu_o + \alpha^2 = \omega^2\varepsilon\mu - k_x^2$$

Even: $\alpha = -\frac{\mu_o}{\mu d}$, $k_x^2 = -\alpha^2 + \omega^2(\varepsilon\mu - \varepsilon_o\mu_o) = \omega^2(\varepsilon\mu - \varepsilon_o\mu_o) - \left(\frac{\mu_o}{\mu d}\right)^2$

c) <u>TM</u>

Odd: $\langle \bar{S} \rangle = \frac{1}{2}\,\mathrm{Re}\,\hat{\bar{E}} \times \hat{\bar{H}}^* = \frac{1}{2}\,\mathrm{Re}[(\hat{E}_x \bar{i}_x + \hat{E}_z \bar{i}_z) \times (\hat{H}^*_y \bar{i}_y)]$

$$= \frac{1}{2}\,\mathrm{Re}[\hat{E}_x \hat{H}^*_y \bar{i}_z - \hat{E}_z \hat{H}^*_y \bar{i}_x]$$

$$= \begin{cases} \dfrac{\omega\varepsilon_o k_z}{\alpha^2}|A_2|^2 e^{-2\alpha(x-d)}\bar{i}_z & x > d \\[2ex] \dfrac{\omega\varepsilon k_z}{k_x^2}|A_1|^2 \cos^2 k_x x\, \bar{i}_z & |x| < d \\[2ex] \dfrac{\omega\varepsilon_o k_z}{\alpha^2}|A_3|^2 e^{2\alpha(x+d)}\bar{i}_z & x < -d \end{cases}$$

$$\langle P_z \rangle = \int_{-\infty}^{-d} \frac{\omega\varepsilon_o k_z}{\alpha^2}|A_3|^2 e^{2\alpha(x+d)}dx + \int_{-d}^{d}\frac{\omega\varepsilon k_z}{k_x^2}|A_1|^2\cos^2 k_x x\, dx$$

$$+ \int_{d}^{\infty}\frac{\omega\varepsilon_o k_z}{\alpha^2}|A_2|^2 e^{-2\alpha(x-d)}dx$$

$$= \frac{\omega\varepsilon_o k_z}{2\alpha^3}\left[|A_3|^2 + |A_2|^2\right] + \frac{\omega\varepsilon k_z}{k_x^3}|A_1|^2\left[k_x d + \frac{1}{2}\sin 2k_x d\right]$$

$$|A_3| = |A_2| = |A_1|\sin k_x d = |A_1|\frac{\varepsilon\alpha}{\varepsilon_o k_x}\cos k_x d$$

$$\langle P_z\rangle = |A_1|^2\left[\frac{\omega\varepsilon_o k_z}{\alpha^3}\sin^2 k_x d + \frac{\omega\varepsilon k_z}{k_x^3}\left(k_x d + \frac{1}{2}\sin 2k_x d\right)\right]$$

Even: $$\langle\overline{S}\rangle = \frac{1}{2}\mathrm{Re}\,\hat{\overline{E}} \times \hat{\overline{H}}^* = \frac{1}{2}\mathrm{Re}\left[(\hat{E}_x\overline{i}_x + \hat{E}_z\overline{i}_z) \times \hat{H}_y^*\overline{i}_y\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\hat{E}_x\hat{H}_y^*\overline{i}_z - \hat{E}_z\hat{H}_y^*\overline{i}_x\right]$$

$$= \begin{cases} \dfrac{\omega\varepsilon_o k_z}{\alpha^2}|B_2|^2\, e^{-2\alpha(x-d)}\overline{i}_z & x > d \\[2ex] \dfrac{\omega\varepsilon k_z}{k_x^2}|B_1|^2\sin^2 k_x x\,\overline{i}_z & -d < x < d \\[2ex] \dfrac{\omega\varepsilon_o k_z}{\alpha^2}|B_3|^2\, e^{2\alpha(x-d)}\overline{i}_z & x > d \end{cases}$$

$$\langle P_z\rangle = \frac{\omega\varepsilon_o k_z}{2\alpha^3}\left[|B_3|^2 + |B_2|^2\right] + \frac{\omega\varepsilon k_z}{k_x^3}|B_1|^2\left[k_x d - \frac{1}{2}\sin 2k_x d\right]$$

$$|B_2| = |B_3| = |B_1|\cos k_x d = \frac{|B_1|\varepsilon\alpha}{\varepsilon_o k_x}\sin k_x d$$

$$\langle P_z\rangle = |B_1|^2\omega k_z\left[\frac{\varepsilon_o}{\alpha^3}\cos^2 k_x d + \frac{\varepsilon}{k_x^3}\left(k_x d - \frac{1}{2}\sin 2k_x d\right)\right]$$

TE

Odd: $$\langle\overline{S}\rangle = \frac{1}{2}\mathrm{Re}(\hat{\overline{E}} \times \hat{\overline{H}}^*) = \frac{1}{2}\mathrm{Re}\left[\hat{E}_y\overline{i}_y \times (\hat{H}_x^*\overline{i}_x + \hat{H}_z^*\overline{i}_z)\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[-\hat{E}_y\hat{H}_x^*\overline{i}_z + \hat{E}_y\hat{H}_z^*\overline{i}_x\right]$$

$$= \begin{cases} \dfrac{\omega\mu_o k_z}{\alpha^2} |A_2|^2 e^{-2\alpha(x-d)}\bar{i}_z & x > d \\ \dfrac{\omega\mu k_z}{k_x^2} |A_1|^2 \cos^2 k_x x\, \bar{i}_z & -d < x < d \\ \dfrac{\omega\mu_o k_z}{\alpha^2} |A_3|^2 e^{2\alpha(x-d)}\bar{i}_z & x < -d \end{cases}$$

$$<P_z> = \frac{\omega\mu_o k_z}{2\alpha^3}[|A_2|^2 + |A_3|^2] + \frac{\omega\mu k_z}{k_x^3}|A_1|^2[k_x d + \frac{1}{2}\sin 2k_x d]$$

$$|A_2| = |A_3| = |A_1|\sin k_x d = \frac{\mu\alpha|A_1|\cos k_x d}{\mu_o k_x}$$

$$<P_z> = |A_1|^2\left[\frac{\omega\mu_o k_z}{\alpha^3}\sin^2 k_x d + \frac{\omega\mu k_z}{k_x^3}[k_x d + \frac{1}{2}\sin 2k_x d]\right]$$

Even:

$$<\bar{S}> = \begin{cases} \dfrac{\omega\mu_o k_z}{\alpha^2} |B_2|^2 e^{-2\alpha(x-d)}\bar{i}_z & x > d \\ \dfrac{\omega\mu k_z}{k_x^2} |B_1|^2 \sin^2 k_x x\, \bar{i}_z & -d < x < d \\ \dfrac{\omega\mu_o k_z}{\alpha^2} |B_3|^2 e^{2\alpha(x+d)}\bar{i}_z & x < -d \end{cases}$$

$$<P_z> = \frac{\omega\mu_o k_z}{2\alpha^3}[|B_2|^2 + |B_3|^2] + \frac{\omega\mu k_z}{k_x^3}|B_1|^2[k_x d - \frac{1}{2}\sin 2k_x d]$$

$$|B_2| = |B_3| = |B_1|\cos k_x d = \frac{\mu\alpha}{\mu_o k_x}|B_1|\sin k_x d$$

$$<P_z> = |B_1|^2\left[\frac{\omega\mu_o k_z}{\alpha^3}\cos^2 k_x d + \frac{\omega\mu k_z}{k_x^3}[k_x d - \frac{1}{2}\sin 2k_x d]\right]$$

d) $<P_d> = \frac{1}{2}\sigma|\hat{E}|^2 \qquad -d < x < d$

$$\alpha_z = \frac{1}{2}\frac{<P_d>}{<P_z>}\ ; \quad <P_d> = \int_{-d}^{+d}\frac{1}{2}\sigma|\hat{E}|^2 dx$$

TM

Odd: $\langle p_d \rangle = \frac{1}{2}\sigma|A_1|^2[\sin^2 k_x x + (\frac{k_z}{k_x})^2 \cos^2 k_x x]$

$$\langle P_d \rangle = \frac{1}{2}\sigma \frac{|A_1|^2}{k_x}[k_x d(1+(\frac{k_z}{k_x})^2) - \frac{1}{2}\sin 2k_x d(1-(\frac{k_z}{k_x})^2)]$$

$$\alpha_z = \frac{\sigma}{4\omega k_z k_x} \frac{[d(1+(\frac{k_z}{k_x})^2) - \frac{1}{2}\sin 2k_x d(1-(\frac{k_z}{k_x})^2)]}{[\frac{\varepsilon_o}{\alpha^3}\sin^2 k_x d + \frac{\varepsilon}{k_x^3}(k_x d + \frac{1}{2}\sin 2k_x d)]}$$

Even: $\langle p_d \rangle = \frac{1}{2}\sigma|B_1|^2[\cos^2 k_x x + (\frac{k_z}{k_x})^2 \sin^2 k_x x]$

$$\langle P_d \rangle = \frac{1}{2}\sigma \frac{|B_1|^2}{k_x}[k_x d(1+(\frac{k_z}{k_x})^2) + \frac{1}{2}\sin 2k_x d(1-(\frac{k_z}{k_x})^2)]$$

TE

Odd: $\langle p_d \rangle = \frac{\sigma}{2}(\frac{\omega\mu}{k_x})^2|A_1|^2 \cos^2 k_x x$

$$\langle P_d \rangle = \frac{\sigma}{2}(\frac{\omega\mu}{k_x})^2 \frac{|A_1|^2}{k_x}[k_x d + \frac{1}{2}\sin 2k_x d]$$

Even: $\langle p_d \rangle = \frac{\sigma}{2}(\frac{\omega\mu}{k_x})^2|B_1|^2 \sin^2 k_x x$

$$\langle P_d \rangle = \frac{\sigma}{2}(\frac{\omega\mu}{k_x})^2 \frac{|B_1|^2}{k_x}[k_x d - \frac{1}{2}\sin 2k_x d]$$

33. a) $E_y(x=0) = E_z(x=0) = 0$

TM even not allowed

TE odd not allowed

Allowed modes: TM odd, TE even

b) TM odd

$$\hat{\sigma}_f(x=0) = \varepsilon\hat{E}_x(x=0) = -\frac{j\varepsilon k_z A_1}{k_x}$$

$$\hat{K}_z(x=0) = \hat{H}_y(x=0) = -\frac{j\omega\varepsilon A_1}{k_x}$$

<u>TE even</u>

$$\hat{\sigma}_f(x=0) = 0$$

$$\hat{K}_y(x=0) = B_1$$

c) TM odd: $\nabla_\Sigma \cdot \overline{K} + \frac{\partial\sigma_f}{\partial t} \rightarrow -jk_x\hat{K}_z + j\omega\hat{\sigma}_f = -jk_z\left(\frac{-j\omega\varepsilon A_1}{k_x}\right) + j\omega\left(\frac{-j\varepsilon k_z A_1}{k_x}\right) = 0$

TE even: $\nabla_\Sigma \cdot \overline{K} = 0$

d) $$\langle P_d\rangle = \frac{|\overline{K}|^2}{\sigma_w\delta} = \begin{cases} \left|\frac{\omega\varepsilon A_1}{k_x}\right|^2 \frac{1}{\sigma_w\delta} & \text{TM} \\ \frac{|B_1|^2}{\sigma_w\delta} & \text{TE} \end{cases}$$

e) $$\alpha_z = \frac{1}{2}\frac{\langle P_d\rangle}{\langle P_z\rangle}$$

TM odd: $$\langle S_z\rangle = \begin{cases} \frac{\omega\varepsilon_o k_z}{\alpha^2}|A_2|^2 e^{-2\alpha(x-d)} = \frac{\omega\varepsilon^2 k_z}{\varepsilon_o k_x^2}|A_1|^2\cos^2 k_x d\, e^{-2\alpha(x-d)} & x > d \\ \frac{\omega\varepsilon k_z}{k_x^2}|A_1|^2\cos^2 k_x x & 0 < x < d \end{cases}$$

TE even: $$\langle S_z\rangle = \begin{cases} \frac{\omega\mu_o k_z}{\alpha^2}|B_2|^2 e^{-2\alpha(x-d)} = \frac{\omega\mu^2 k_z}{\mu_o k_x^2}|B_1|^2\sin^2 k_x d\, e^{-2\alpha(x-d)} & x > d \\ \frac{\omega\mu k_z}{k_x^2}|B_1|^2\sin^2 k_x x & 0 < x < d \end{cases}$$

<u>TM</u>

$$\langle P_z\rangle = \frac{\omega\varepsilon k_z}{k_x^2}|A_1|^2\left\{\int_0^d \cos^2 k_x x\,dx + \int_d^\infty \frac{\varepsilon}{\varepsilon_o}\cos^2 k_x d\, e^{-2\alpha(x-d)}dx\right\}$$

$$= \frac{\omega\varepsilon k_z}{k_x^3}|A_1|^2\left\{\frac{1}{2}k_x d + \frac{1}{4}\sin 2k_x d + \frac{\varepsilon}{\varepsilon_o}\frac{\cos^2 k_x d}{2\alpha}\right\}$$

$$\alpha_z = \frac{\omega\varepsilon k_x}{\sigma_w \delta k_z} \left[k_x d + \frac{1}{2}\sin 2k_x d + \frac{\varepsilon}{\varepsilon_o \alpha}\cos^2 k_x d\right]^{-1}$$

<u>TE</u>

$$\langle P_z \rangle = \frac{\omega\mu k_z}{k_x^2} |B_1|^2 \left\{\int_0^d \sin^2 k_x x dx + \int_d^\infty \frac{\mu}{\mu_o}\sin^2 k_x d e^{-2\alpha(x-d)} dx\right\}$$

$$= \frac{\omega\mu k_z}{k_x^3} |B_1|^2 \left\{\frac{1}{2}k_x d - \frac{1}{4}\sin 2k_x d + \frac{\mu}{\mu_o}\frac{\sin^2 k_x d}{2\alpha}\right\}$$

$$\alpha_z = \frac{k_x^3}{\sigma_w \delta\omega\mu k_z}\left[k_x d - \frac{1}{2}\sin 2k_x d + \frac{\mu}{\mu_o \alpha}\sin^2 k_x d\right]^{-1}$$

CHAPTER 9

RADIATION

Section 9.1

1. a) $\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$

$$\nabla \times \overline{H} = \overline{J}_s + \sigma\overline{E} + \varepsilon\frac{\partial \overline{E}}{\partial t}$$

$$\nabla \cdot \overline{B} = 0 \rightarrow \overline{B} = \nabla \times \overline{A}$$

$$\nabla \cdot \overline{E} = \frac{\rho}{\varepsilon}$$

b) $\nabla \times (\overline{E} + \frac{\partial \overline{A}}{\partial t}) = 0 \rightarrow \overline{E} = -\frac{\partial \overline{A}}{\partial t} - \nabla V$

$$\nabla\times\overline{H} = \frac{1}{\mu}\nabla\times(\nabla\times\overline{A}) = \frac{1}{\mu}[\nabla(\nabla\cdot\overline{A}) - \nabla^2\overline{A}] = \overline{J}_s - \sigma[\frac{\partial \overline{A}}{\partial t} + \nabla V] - \varepsilon\frac{\partial^2 \overline{A}}{\partial t^2} - \varepsilon\nabla\frac{\partial V}{\partial t}$$

$$\nabla^2\overline{A} - \nabla[\nabla\cdot\overline{A} + \mu\sigma V + \mu\varepsilon\frac{\partial V}{\partial t}] - \mu\sigma\frac{\partial \overline{A}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \overline{A}}{\partial t^2} = -\mu\overline{J}_s$$

Gauge condition: $\nabla\cdot\overline{A} + \mu\sigma V + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

$$\nabla^2\overline{A} - \mu\sigma\frac{\partial \overline{A}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \overline{A}}{\partial t^2} = -\mu\overline{J}_s$$

$$\nabla\cdot\overline{E} = -\frac{\partial}{\partial t}\nabla\cdot\overline{A} - \nabla^2 V = \frac{\rho}{\varepsilon}$$

$$\nabla^2 V - \mu\sigma\frac{\partial V}{\partial t} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

c) $V = \mathrm{Re}\,\hat{V}(r)e^{j\omega t}, \; \rho = 0$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\hat{V}}{dr}) - j\omega\mu\sigma\hat{V} + \frac{\omega^2}{c^2}\hat{V} = 0$$

$$\frac{d^2}{dr^2}(r\hat{V}) + (\frac{\omega^2}{c^2} - j\omega\mu\sigma)r\hat{V} = 0$$

$$\hat{V} = \frac{\hat{Q}}{4\pi\varepsilon r} e^{-j\sqrt{\frac{\omega^2}{c^2} - j\omega\mu\sigma}\, r}$$

d) $\nabla \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t}$

$$\nabla \times \frac{\partial \overline{H}}{\partial t} = \frac{\partial \overline{J}_s}{\partial t} + \omega_p^2 \varepsilon \overline{E} + \varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}$$

$$\nabla \cdot \overline{B} = 0 \rightarrow \overline{B} = \nabla \times \overline{A}$$

$$\overline{E} = -\frac{\partial \overline{A}}{\partial t} - \nabla V$$

$$\nabla\times(\nabla\times\frac{\partial \overline{A}}{\partial t}) = \nabla(\nabla\cdot\frac{\partial \overline{A}}{\partial t}) - \nabla^2\frac{\partial \overline{A}}{\partial t} = \mu\frac{\partial \overline{J}_s}{\partial t} - \frac{\omega_p^2}{c^2}(\frac{\partial \overline{A}}{\partial t} + \nabla V) - \frac{1}{c^2}\frac{\partial^3 \overline{A}}{\partial t^3} - \frac{1}{c^2}\nabla\frac{\partial^2 V}{\partial t^2}$$

$$\nabla^2\frac{\partial \overline{A}}{\partial t} - \nabla[\nabla\cdot\frac{\partial \overline{A}}{\partial t} + \frac{\omega_p^2}{c^2}V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}] - \frac{\omega_p^2}{c^2}\frac{\partial \overline{A}}{\partial t} - \frac{1}{c^2}\frac{\partial^3 \overline{A}}{\partial t^3} = -\mu\frac{\partial \overline{J}_s}{\partial t}$$

Gauge condition: $\nabla \cdot \dfrac{\partial \overline{A}}{\partial t} + \dfrac{\omega_p^2}{c^2} V + \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2} = 0$

$$\nabla^2\overline{A} - \frac{\omega_p^2}{c^2}\overline{A} - \frac{1}{c^2}\frac{\partial^2 \overline{A}}{\partial t^2} = -\mu\overline{J}_s$$

$$\nabla\cdot\overline{E} = -\nabla\cdot\frac{\partial \overline{A}}{\partial t} - \nabla^2 V = \frac{\rho}{\varepsilon}$$

$$\nabla^2 V - \frac{\omega_p^2}{c^2}V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

For $\rho = 0$ with $Q(r{=}0,t) = \mathrm{Re}\,\hat{Q}e^{j\omega t}$

$$\frac{d^2}{dr^2}(r\hat{V}) + \frac{(\omega^2 - \omega_p^2)}{c^2} r\hat{V} = 0$$

$$\hat{V} = \frac{\hat{Q}\, e^{-j\frac{r}{c}\sqrt{\omega^2 - \omega_p^2}}}{4\pi\varepsilon r}$$

2\. a) $\nabla^2\overline{A} - \dfrac{1}{c^2}\dfrac{\partial^2 \overline{A}}{\partial t^2} = -\mu\overline{J}_f = 0$

$$\overline{A} = \mathrm{Re}\,\hat{A}_x e^{j(\omega t - k_x x - k_z z)}\,\overline{i}_x \qquad z > 0$$

$$\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial z^2} + \frac{\omega^2}{c^2} A_x = 0 \rightarrow k_x^2 + k_z^2 = \frac{\omega^2}{c^2}$$

$$\nabla \cdot \overline{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \rightarrow -jk_x \hat{A}_x = -\frac{j\omega}{c^2}\hat{V} \rightarrow \hat{V} = \frac{k_x c^2}{\omega}\hat{A}_x$$

(b) $$\overline{E} = -\frac{\partial \overline{A}}{\partial t} - \nabla V \rightarrow \hat{\overline{E}} = -j\omega \hat{A}_x \overline{i}_x + (jk_x \overline{i}_x + jk_z \overline{i}_z)\hat{V}$$

$$= \overline{i}_x j\left(\frac{k_x^2 c^2 - \omega^2}{\omega}\right)\hat{A}_x + \overline{i}_z \frac{jk_z k_x c^2}{\omega}\hat{A}_x$$

$$\overline{B} = \nabla \times \overline{A} = \overline{i}_y \frac{\partial A_x}{\partial z} \rightarrow \hat{B}_y = -jk_z \hat{A}_x$$

$$\hat{H}_y = -\frac{jk_z}{\mu}\hat{A}_x \rightarrow \hat{H}_y(z=0) = -\frac{K_o}{2} = -\frac{jk_z}{\mu}\hat{A}_x \rightarrow \hat{A}_x = \frac{\mu K_o}{2jk_z}$$

For $z < 0$ let $k_z \rightarrow -k_z$.

(c) $$\nabla^2 A_x - \frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} = -\mu J_o e^{j(\omega t - k_x x)} \qquad |z| < a$$

$$A_x = \mathrm{Re}\,\hat{A}_x(z) e^{j(\omega t - k_x x)}$$

$$\frac{d^2 \hat{A}_x}{dz^2} - \left(k_x^2 - \frac{\omega^2}{c^2}\right)\hat{A}_x = -\mu J_o$$

$$\hat{A}_x = \mathrm{Re}\left\{\underbrace{\left(\frac{\mu J_o}{k_x^2 - \frac{\omega^2}{c^2}}\right) e^{j(\omega t - k_x x)}}_{\text{Particular Solution}} + \underbrace{A_1 e^{j(\omega t - k_x x - k_z z)} + A_2 e^{j(\omega t - k_x x + k_z z)}}_{\text{Homogeneous Solution}}\right\}$$

$$d\hat{A}_x = \begin{cases} \dfrac{\mu J_o dz'}{2jk_z} e^{j(\omega t - k_x x - k_z (z-z'))} & z > z' \\[2ex] \dfrac{-\mu J_o dz'}{2jk_z} e^{j(\omega t - k_x x + k_z (z-z'))} & z < z' \end{cases}$$

$z > a$

$$\hat{A}_x = \int_{-a}^{a} \frac{\mu J_o dz'}{2jk_z} e^{j(\omega t - k_x x - k_z(z-z'))}$$

$$= \frac{\mu J_o}{2k_z^2} e^{j(\omega t - k_x x - k_z z)} [e^{jk_z a} - e^{-jk_z a}]$$

$$= \frac{\mu J_o j}{k_z^2} \sin k_z a \; e^{j(\omega t - k_x x - k_z z)}$$

$z < -a$

$$\hat{A}_x = \int_{-a}^{a} \frac{-\mu J_o dz'}{2jk_z} e^{j(\omega t - k_x x + k_z(z-z'))}$$

$$= \frac{\mu J_o}{k_z^2} e^{j(\omega t - k_x x + k_z z)} [e^{-jk_z a} - e^{jk_z a}]$$

$$= \frac{-\mu J_o j}{k_z^2} \sin k_z a \; e^{j(\omega t - k_x x + k_z z)}$$

$-a < z < a$

$$\hat{A}_x = \int_{-a}^{z} \frac{-\mu J_o dz'}{2jk_z} e^{j(\omega t - k_x x - k_z(z-z'))} + \int_{z}^{a} \frac{\mu J_o dz'}{2jk_z} e^{j(\omega t - k_x x + k_z(z-z'))}$$

$$= \frac{\mu J_o}{2k_z^2} e^{j(\omega t - k_x x)} [1 - e^{-jk_z(z+a)} + e^{jk_z(z-a)} - 1]$$

$$= \frac{j\mu J_o}{k_z^2} e^{-jk_z a} \sin k_z z \; e^{j(\omega t - k_x x)}$$

$$A_x = \begin{cases} \mathrm{Re}\left[\dfrac{\mu J_o j}{k_z^2} \sin k_z a \; e^{j(\omega t - k_x x - k_z z)}\right] & z > a \\ \mathrm{Re}\left[\dfrac{\mu J_o j}{k_z^2} e^{-jk_z a} \sin k_z z \; e^{j(\omega t - k_x x)}\right] & |z| < a \\ \mathrm{Re}\left[\dfrac{-\mu J_o j}{k_z^2} \sin k_z a \; e^{j(\omega t - k_x x + k_z z)}\right] & z < -a \end{cases}$$

$$V = \frac{k_x c^2}{\omega} A_x$$

$$\overline{E} = -\frac{\partial \overline{A}}{\partial t} - \nabla V$$

$$\hat{E}_x = -j\omega\hat{A}_x + jk_x\hat{V} = \hat{A}_x(-j\omega + \frac{jk_x^2c^2}{\omega}) = \frac{j\hat{A}_x}{\omega}(k_x^2c^2 - \omega^2)$$

$$\hat{E}_z = jk_z\hat{V} = \frac{jk_zk_xc^2}{\omega}\hat{A}_x$$

$$H_y = \frac{1}{\mu}\frac{\partial A_x}{\partial z} \rightarrow \hat{H}_y = \frac{-jk_z}{\mu}\hat{A}_x$$

3. (a)

$$V = \int_S \frac{\sigma_f(t - \frac{r_{QP}}{c})}{4\pi\varepsilon r_{QP}} R^2\sin\theta d\theta d\phi$$

$$r_{QP}^2 = [r^2 + R^2 - 2rR\cos\theta]$$

$$\hat{V} = \frac{\sigma_o R^2}{2\varepsilon}\int_{\theta=0}^{\pi}\frac{e^{-j\frac{\omega}{c}r_{QP}}}{r_{QP}}\sin\theta d\theta$$

$$2r_{QP}dr_{QP} = 2rR\sin\theta d\theta \rightarrow \frac{\sin\theta d\theta}{r_{QP}} = \frac{dr_{QP}}{rR}$$

$$\hat{V} = \frac{\sigma_o R}{2\varepsilon r}\int_{|r-R|}^{r+R} e^{-j\frac{\omega}{c}r_{QP}}dr_{QP}$$

$$= \frac{\sigma_o R}{2\varepsilon r(\frac{-j\omega}{c})}[e^{-j\frac{\omega}{c}(r+R)} - e^{-j\frac{\omega}{c}|r-R|}]$$

$r < R$

$$e^{-j\frac{\omega}{c}(r+R)} - e^{-j\frac{\omega}{c}(R-r)} = e^{-j\frac{\omega}{c}R}[e^{-j\frac{\omega}{c}r} - e^{j\frac{\omega}{c}r}]$$

$$\hat{V} = \frac{j\sigma_o Rce^{-j\frac{\omega}{c}R}}{2\varepsilon r\omega}[e^{-j\frac{\omega}{c}r} - e^{j\frac{\omega}{c}r}]$$

$r > R$

$$e^{-j\frac{\omega}{c}(r+R)} - e^{-j\frac{\omega}{c}(r-R)} = e^{-j\frac{\omega}{c}r}[e^{-j\frac{\omega}{c}R} - e^{j\frac{\omega}{c}R}]$$

$$\hat{V} = \frac{j\sigma_o Rce^{-j\frac{\omega}{c}r}}{2\varepsilon r\omega}\left[e^{-j\frac{\omega}{c}R} - e^{j\frac{\omega}{c}R}\right]$$

$$\nabla\cdot\overline{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\hat{A}_r) = \frac{-j\omega}{c^2}\hat{V}$$

$$\frac{d}{dr}(r^2\hat{A}_r) = \frac{\sigma_o Rre^{-j\omega r/c}}{2\varepsilon c}\left[e^{-j\frac{\omega}{c}R} - e^{j\frac{\omega}{c}R}\right]$$

$$\hat{A}_r = \frac{\sigma_o R}{2\varepsilon cr^2}\left[e^{-j\frac{\omega}{c}R} - e^{j\frac{\omega}{c}R}\right]\frac{e^{-j\omega r/c}}{\left(\frac{\omega^2}{c^2}\right)}\left(\frac{j\omega r}{c} + 1\right)$$

$$= \frac{\sigma_o Rc}{2\varepsilon\omega^2 r^2}\left(\frac{j\omega r}{c} + 1\right)e^{-j\omega r/c}\left[e^{-j\omega R/c} - e^{j\omega R/c}\right] + \frac{C_1}{r^2}$$

$r < R$

$$\frac{d}{dr}(r^2\hat{A}_r) = \frac{\sigma_o Rr}{2\varepsilon c}e^{-j\frac{\omega}{c}R}\left[e^{-j\frac{\omega}{c}r} - e^{j\frac{\omega}{c}r}\right]$$

$$\hat{A}_r = \frac{\sigma_o Rc}{2\varepsilon r^2}\frac{e^{-j\omega R/c}}{\omega^2}\left[e^{-j\frac{\omega}{c}r}\left(\frac{j\omega r}{c} + 1\right) + e^{j\frac{\omega}{c}r}\left(\frac{j\omega r}{c} - 1\right)\right] + \frac{C_2}{r^2}$$

(b) $$E_r = -\frac{\partial A_r}{\partial t} - \frac{dV}{dr}$$

$r > R$

$$\hat{E}_r = \frac{-j\sigma_o Rc}{2\varepsilon r^2\omega}\left(\frac{j\omega r}{c} + 1\right)e^{-j\omega r/c}\left[e^{-j\omega R/c} - e^{j\omega R/c}\right][1 - 1] - \frac{j\omega C_1}{r^2}$$

$$= \frac{-j\omega C_1}{r^2}$$

$r < R$

$$\hat{E}_r = \frac{-j\sigma_o Rc}{2\varepsilon r^2\omega}e^{-j\omega R/c}\left[e^{-j\frac{\omega}{c}r}\left(\frac{j\omega r}{c} + 1\right)(1 - 1) + e^{j\omega r/c}\left(\frac{j\omega r}{c} - 1\right)(1 - 1)\right] - \frac{j\omega C_2}{r^2}$$

$$= \frac{-j\omega C_2}{r^2}$$

$$\hat{E}_r(r=0) = \text{finite} \rightarrow C_2 = 0$$

$$\hat{E}_r(r=R_+) = \frac{\sigma_o}{\varepsilon} = \frac{-j\omega C_1}{R^2} \rightarrow C_1 = \frac{j\sigma_o R^2}{\varepsilon\omega}$$

$$\hat{E}_r = \begin{cases} \dfrac{\sigma_o R^2}{\varepsilon r^2} & r > R \\ 0 & r < R \end{cases}$$

$$\overline{H} = \frac{1}{\mu} \nabla \times \overline{A} = 0$$

(c) $\nabla \times \overline{H} = \overline{J} + \varepsilon \dfrac{\partial \overline{E}}{\partial t} \rightarrow \hat{J}_r = -j\omega\varepsilon\hat{E}_r$

Section 9.2

4. $d\ell_{eff} = \dfrac{1}{I_o} \displaystyle\int_{-d\ell/2}^{d\ell/2} \hat{I}(z)dz$, $R = \dfrac{2\pi\eta}{3} \left(\dfrac{d\ell_{eff}}{\lambda}\right)^2$

$$\hat{\lambda} = -\frac{1}{j\omega}\frac{d\hat{I}}{dz}$$

(a) $\hat{I}(z) = I_o \cos\alpha z$

$$d\ell_{eff} = \int_{-d\ell/2}^{d\ell/2} \cos\alpha z dz = \frac{2\sin\alpha \frac{d\ell}{2}}{\alpha}$$

$$\hat{\lambda}(z) = \frac{I_o\alpha}{j\omega} \sin\alpha z$$

(b) $\hat{I}(z) = I_o e^{-\alpha|z|}$

$$d\ell_{eff} = \int_{-d\ell/2}^{0} e^{\alpha z}dz + \int_{0}^{d\ell/2} e^{-\alpha z}dz$$

$$= \frac{2}{\alpha}(1 - e^{-\alpha d\ell/2})$$

$$\hat{\lambda}(z) = \begin{cases} \dfrac{I_o\alpha}{j\omega} e^{-\alpha z} & z > 0 \\ \dfrac{-\alpha I_o}{j\omega} e^{\alpha z} & z < 0 \end{cases}$$

(c) $\hat{I}(z) = I_o \cosh\alpha z$

$$d\ell_{eff} = \int_{-d\ell/2}^{d\ell/2} \cosh\alpha z dz = \frac{2}{\alpha} \sinh \frac{\alpha d\ell}{2}$$

$$\hat{\lambda}(z) = \frac{-\alpha I_o}{j\omega} \sinh\alpha z$$

5. $<\overline{S}> = \frac{1}{2} \text{Re } \hat{\overline{E}} \times \hat{\overline{H}}^*$

$$= \frac{1}{2} \frac{|m|^2 k^4 \eta}{(4\pi)^2 r^2} \sin^2\theta \, \overline{i}_r$$

$$<P> = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} <S_r> r^2 \sin\theta d\theta d\phi$$

$$= \frac{1}{2} \frac{|m|^2 k^4 \eta 2\pi}{(4\pi)^2} \int_{\theta=0}^{\pi} \sin^3\theta d\theta$$

$$= \frac{|m|^2 k^4 \eta}{12\pi} = |I|^2 \frac{(dS)^2 k^4 \eta}{12\pi}$$

$$<P> = \frac{1}{2} |I|^2 R \rightarrow R = \frac{k^2 \eta (dS)^2}{6\pi}$$

6. (a) $(\hat{p}_z)_{ind} = 4\pi\varepsilon_o R^3 \hat{E}_o$

(b) $$(\hat{p}_z)_{ind} = \frac{4\pi\varepsilon_o(\varepsilon - \varepsilon_o) R^3 \hat{E}_o}{2\varepsilon_o + \varepsilon}$$

(c) $$<P> = \frac{\omega^4 |\hat{p}_{ind}|^2 \eta}{12\pi c^2}$$

7. (a)

$$\overline{H}_{dipole} = \frac{m}{4\pi r^3}\left[2\cos\theta\,\overline{i}_r + \sin\theta\,\overline{i}_\theta\right] \qquad \text{(from Eq. (8) Section (5.5.1))}$$

$$\overline{H}_{sphere} = H_o\left\{\left(1-\frac{R^3}{r^3}\right)\cos\theta\,\overline{i}_r - \left(1+\frac{R^3}{2r^3}\right)\sin\theta\,\overline{i}_\theta\right\} \qquad \text{(from Eq. (15) Section (5.7.2ii))}$$

$\overline{H}_{sphere}$ is sum of uniform field plus equivalent induced magnetic dipole moment.

$$H_o R^3 = \frac{-2m_{ind}}{4\pi} \rightarrow m_{ind} = -2\pi H_o R^3$$

(b) $$\hat{\overline{E}} = \frac{\hat{m}_{ind} jk^3}{4\pi}\,\eta e^{-jkr}\sin\theta\left[\frac{1}{(jkr)} + \frac{1}{(jkr)^2}\right]\overline{i}_\phi$$

$$\hat{\overline{H}} = \frac{-j\hat{m}_{ind}}{4\pi}k^3 e^{-jkr}\left\{\overline{i}_r 2\cos\theta\left(\frac{1}{(jkr)^2} + \frac{1}{(jkr)^3}\right) + \overline{i}_\theta \sin\theta\left(\frac{1}{(jkr)} + \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3}\right)\right\}$$

(c) From Prob. 5, $$\langle P\rangle = \frac{|\hat{m}_{ind}|^2 k^4 \eta}{12\pi}$$

$$\langle P\rangle \propto \omega^4$$

8. (a) $$\overline{B} = \nabla\times\overline{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\overline{i}_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\overline{i}_\theta$$

$$\frac{dr}{r\,d\theta} = \frac{B_r}{B_\theta} = \frac{-1}{\sin\theta}\,\frac{\frac{\partial}{\partial\theta}(A_\phi\sin\theta)}{\frac{\partial}{\partial r}(rA_\phi)}$$

$$\frac{\partial}{\partial r}(r\sin\theta A_\phi)dr + \frac{\partial}{\partial\theta}(r\sin\theta A_\phi)\,d\theta = 0$$

$$r\sin\theta A_\phi = \text{constant}$$

(b) $$A_\phi = \mathrm{Re}\left[\hat{A}_\phi e^{j\omega t}\right] = \mathrm{Re}\left[\frac{\mu\hat{m}}{4\pi r^2}\sin\theta(1+jkr)e^{j(\omega t-kr)}\right]$$

Assume $\hat{m} = m$ real

$$A_\phi = \frac{\mu m}{4\pi}\frac{\sin\theta}{r^2}\left[\cos(\omega t - kr) - kr\sin(\omega t - kr)\right]$$

Magnetic field lines $\rightarrow \sin^2\theta\left[\frac{\cos(\omega t-kr)}{kr} - \sin(\omega t-kr)\right] = \text{constant}$

Section 9.3

9. (a) $r_1 \approx r - a\cos\theta, \quad r_2 \approx r + a\cos\theta$

$$\hat{E}_\theta \approx \frac{\hat{E}_o}{jkr}\sin\theta e^{-jkr}\left[e^{jka\cos\theta} + e^{-j(ka\cos\theta-\chi)}\right]$$

$$\approx \frac{2\hat{E}_o}{jkr}\sin\theta e^{-j(kr-\frac{\chi}{2})}\left[\cos(ka\cos\theta - \frac{\chi}{2})\right]$$

$$\hat{H}_\phi = \frac{\hat{E}_\theta}{\eta}$$

(b) $$\langle S_r\rangle = \frac{1}{2}\frac{|\hat{E}_\theta|^2}{\eta} = \frac{2|\hat{E}_o|^2}{k^2r^2\eta}\sin^2\theta\cos^2(ka\cos\theta - \frac{\chi}{2})$$

(c) $\langle S_r\rangle = 0$ when $ka\cos\theta - \frac{\chi}{2} = (2n+1)\frac{\pi}{2}$

$$\theta = \cos^{-1}\left[\frac{(2n+1)\frac{\pi}{2} + \frac{\chi}{2}}{ka}\right]$$

$\langle S_r\rangle_{max}$ when $ka\cos\theta - \frac{\chi}{2} = n\pi$

$$\theta = \cos^{-1}\left[\frac{n\pi + \frac{\chi}{2}}{ka}\right]$$

(d) $2a = \frac{\lambda}{2} \rightarrow ka = \frac{\pi}{2}$

Broadside $\rightarrow \theta = 90° \rightarrow \chi = \pm 2n\pi$

Endfire $\rightarrow \theta = 0 \rightarrow \langle S_r\rangle = 0$ no endfire

(e) $$\hat{E}_\theta = \frac{\hat{E}_o}{jkr}\sin\theta e^{-jkr}\sum_{n=-N}^{N} e^{jn(ka\cos\theta - \chi_o)}$$

$$= \frac{\hat{E}_o}{jkr}\sin\theta e^{-jkr}\left\{\frac{\sin(N+\frac{1}{2})(ka\cos\theta-\chi_o)}{\sin\frac{1}{2}(ka\cos\theta-\chi_o)}\right\}$$

$$\hat{H}_\phi = \frac{\hat{E}_\theta}{\eta}$$

$$<S_r> = \frac{1}{2}\frac{|\hat{E}_\theta|^2}{\eta} = \frac{|\hat{E}_o|^2}{k^2r^2\eta}\sin^2\theta\left\{\frac{\sin(N+\frac{1}{2})(ka\cos\theta-\chi_o)}{\sin\frac{1}{2}(ka\cos\theta-\chi_o)}\right\}^2$$

$$<S_r> = 0 \rightarrow (N+\frac{1}{2})(ka\cos\theta-\chi_o) = n\pi$$

$$\theta = \cos^{-1}\frac{\frac{n\pi}{N+\frac{1}{2}}+\chi_o}{ka}$$

$$<S_r>_{max} \rightarrow (N+\frac{1}{2})(ka\cos\theta-\chi_o) = (2n+1)\frac{\pi}{2}$$

$$\theta = \cos^{-1}\left[\frac{\frac{(2n+1)(\pi/2)}{N+\frac{1}{2}}+\chi_o}{ka}\right]$$

10. (a) $\hat{E}_\theta = \frac{\sin\theta e^{-jkr}}{jkr}[\hat{E}_1 e^{jka\cos\theta} + \hat{E}_2 + \hat{E}_3 e^{-jka\cos\theta}]$

$$\hat{H}_\phi = \frac{\hat{E}_\theta}{\eta}$$

(b) $<S_r> = \frac{1}{2}\frac{|\hat{E}_\theta|^2}{\eta}$

(c) (i) $\hat{I}_1 = \hat{I}_3 = I_o,\ \hat{I}_2 = 2I_o \rightarrow \hat{E}_1 = \hat{E}_3 \equiv \hat{E}_o,\ \hat{E}_2 = 2\hat{E}_o$

$$\hat{E}_\theta = \frac{2\hat{E}_o\sin\theta e^{-jkr}}{jkr}\left[\cos(ka\cos\theta)+1\right] = \frac{4\hat{E}_o}{jkr}\sin\theta e^{-jkr}\cos^2(\frac{ka\cos\theta}{2})$$

$$<S_r> = 0 \rightarrow \theta = 0,\ 180°;\ \theta = \cos^{-1}[\frac{(2n+1)\pi}{ka}]$$

$$<S_r>_{max} \rightarrow \theta = \cos^{-1}[\frac{2n\pi}{ka}]$$

(ii) $\hat{I}_1 = \hat{I}_3 = I_o,\ \hat{I}_2 = -2I_o,\ \hat{E}_2 = -2\hat{E}_o,\ \hat{E}_1 = \hat{E}_3 = \hat{E}_o$

$$\hat{E}_\theta = \frac{2\hat{E}_o}{jkr}\sin\theta e^{-jkr}[\cos(ka\cos\theta) - 1] = \frac{-4\hat{E}_o\sin\theta e^{-jkr}}{jkr}\sin^2(\frac{ka\cos\theta}{2})$$

$$<S_r> = 0 \rightarrow \theta = 0,\ 180°;\ \theta = \cos^{-1}[\frac{2n\pi}{ka}]$$

$$<S_r>_{max} \rightarrow \theta = \cos^{-1}[\frac{(2n+1)\pi}{ka}]$$

(iii) $\hat{I}_1 = -\hat{I}_3 = jI_o,\ \hat{I}_2 = I_o \rightarrow \hat{E}_1 = -\hat{E}_3 = j\hat{E}_o,\ \hat{E}_2 = \hat{E}_o$

$$\hat{E}_\theta = \frac{\hat{E}_o\sin\theta e^{-jkr}}{jkr}[e^{jka\cos\theta} - e^{-jka\cos\theta} + 2j]$$

$$= \frac{2\hat{E}_o\sin\theta e^{-jkr}}{kr}[1 + \sin(ka\cos\theta)]$$

$$<S_r> = 0 \rightarrow \theta = 0,\ 180\ ;\ \theta = \cos^{-1}[\frac{(4n+3)\pi}{2ka}]$$

$$<S_r>_{max} \rightarrow \theta = \cos^{-1}[\frac{(4n+1)\pi}{2ka}]$$

11. (a) $dI = K_o dx$

$$d\hat{E}_\theta = \frac{E'_o\sin\theta e^{-jkr}}{jkr}e^{jkx\sin\theta\cos\phi}dx;\ E'_o = \frac{-K_o d\ell k^2\eta}{4\pi}$$

$$\hat{E}_\theta = \frac{E'_o\sin\theta e^{-jkr}}{jkr}\int_{-L/2}^{+L/2} e^{jkx\sin\theta\cos\phi}dx$$

$$= \frac{E'_o\sin\theta e^{-jkr}}{(jkr)jk\sin\theta\cos\phi}[e^{\frac{jkL}{2}\sin\theta\cos\phi} - e^{\frac{-jkL}{2}\sin\theta\cos\phi}]$$

$$= \frac{2jE'_o e^{-jkr}}{-k^2 r\cos\phi}[\sin(\frac{kL}{2}\sin\theta\cos\phi)]$$

$$\hat{H}_\phi = \frac{\hat{E}_\theta}{\eta}$$

(b) $<S_r> = 0 \qquad \frac{kL}{2}\sin\theta\cos\phi = n\pi$

$$\langle S_r\rangle_{max} \qquad \frac{kL}{2}\sin\theta\cos\phi = \frac{(2n+1)\pi}{2}$$

Section 9.4

12. (a) $$\hat{E}_\theta = \eta\hat{H}_\phi = \frac{jk\eta I_o \sin\theta e^{-jkr}}{4\pi r}\left\{\int_{-L/2}^{0}(1+\frac{2z}{L})e^{jkz\cos\theta}dz + \int_{0}^{L/2}(1-\frac{2z}{L})e^{jkz\cos\theta}dz\right\}$$

$$= \frac{jk\eta I_o \sin\theta e^{-jkr}}{4\pi r}\left\{\frac{e^{jkz\cos\theta}}{jk\cos\theta}\left(1+\frac{2}{jkL\cos\theta}(jkz\cos\theta - 1)\right)\Bigg|_{-L/2}^{0}\right.$$

$$\left. + \frac{e^{jkz\cos\theta}}{jk\cos\theta}\left(1-\frac{2}{jkL\cos\theta}(jkz\cos\theta - 1)\right)\Bigg|_{0}^{L/2}\right\}$$

$$= \frac{jk\eta I_o}{4\pi r}\frac{\sin\theta e^{-jkr}}{jk\cos\theta}\left\{1-\frac{2}{jkL\cos\theta} - e^{\frac{-jkL}{2}\cos\theta}\left(1+\frac{2}{jkL\cos\theta}(\frac{-jkL}{2}\cos\theta - 1)\right)\right.$$

$$\left. + e^{\frac{jkL}{2}\cos\theta}\left(1-\frac{2}{jkL\cos\theta}(\frac{jkL}{2}\cos\theta - 1)\right) - \left(1+\frac{2}{jkL\cos\theta}\right)\right\}$$

$$= \frac{k\eta I_o}{4\pi r}\frac{\sin\theta e^{-jkr}}{k\cos\theta}\left\{\frac{-4}{jkL\cos\theta} + \frac{2}{jkL\cos\theta}\left(e^{\frac{jkL}{2}\cos\theta} + e^{\frac{-jkL}{2}\cos\theta}\right)\right\}$$

$$= \frac{\eta I_o \sin\theta e^{-jkr}}{j\pi krL\cos^2\theta}\left[-1 + \cos(\frac{kL}{2}\cos\theta)\right]$$

$$\langle S_r\rangle = \frac{1}{2}\frac{|E_\theta|^2}{\eta}$$

(b) $$\hat{E}_\theta = \eta\hat{H}_\phi = \frac{jk\eta I_o \sin\, e^{-jkr}}{4\pi r}\int_{-L/2}^{L/2}\cos\frac{\pi z}{L}e^{jkz\cos\theta}dz$$

$$= \frac{jk\eta I_o \sin\theta e^{-jkr}}{4\pi r}e^{jkz\cos\theta}\frac{\left[jk\cos\theta\cos\frac{\pi z}{L} + \frac{\pi}{L}\sin\frac{\pi z}{L}\right]}{(\frac{\pi}{L})^2 - (k\cos\theta)^2}\Bigg|_{z=-\frac{L}{2}}^{L/2}$$

$$= \frac{jk\eta I_o}{4rL}\frac{\sin\theta e^{-jkr}}{\left[(\frac{\pi}{L})^2 - (k\cos\theta)^2\right]}\left(e^{\frac{jkL}{2}\cos\theta} + e^{\frac{-jkL}{2}\cos\theta}\right)$$

$$= \frac{jkL\eta I_o}{8r}\sin\theta e^{-jkr}\frac{\cos[\frac{kL}{2}\cos\theta]}{\left[(\frac{kL}{2}\cos\theta)^2 - (\frac{\pi}{2})^2\right]}$$

$$\langle S_r \rangle = \frac{1}{2} \frac{|E_\theta|^2}{\eta}$$

(c) $kL \ll 1$

(a) $$\hat{E}_\theta \approx \frac{\eta I_o \sin\theta e^{-jkr}}{j\pi krL\cos^2\theta} \frac{1}{2} \frac{k^2L^2\cos^2\theta}{4}$$

$$= \frac{\eta I_o \sin\theta kLe^{-jkr}}{j8\pi r} \rightarrow (d\ell)_{eff} = \frac{L}{2}$$

$$R = 40\pi^2(\frac{L}{\lambda})^2$$

(b) $$\hat{E}_\theta \approx \frac{-jkL\eta I_o \sin\theta e^{-jkr}}{2\pi^2 r} \rightarrow (d\ell)_{eff} = \frac{2L}{\pi}$$

$$R = \frac{320L^2}{\lambda^2}$$